Lagrangian spheres, symplectic surfaces and the symplectic mapping class group

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Abstract

Given a Lagrangian sphere in a symplectic 4-manifold (M, ω) with $b^+ = 1$, we find embedded symplectic surfaces intersecting it minimally. When the Kodaira dimension κ of (M, ω) is $-\infty$, this minimal intersection property turns out to be very powerful for both the uniqueness and existence problems of Lagrangian spheres. On the uniqueness side, for a symplectic rational manifold and any class which is not characteristic and ternary, we show that homologous Lagrangian spheres are smoothly isotopic, and when the Euler number is less than 8, we generalize Hind and Evans' Hamiltonian uniqueness in the monotone case. On the existence side, when $\kappa = -\infty$, we give a characterization of classes represented by Lagrangian spheres, which enables us to describe the non-Torelli part of the symplectic mapping class group.

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1 Introduction

For a symplectic 4-manifold (M, ω), symplectic surfaces and Lagrangian surfaces are of complementary dimensions. Thus we can ask what can be said

about their intersection pattern. Welschinger investigated this problem for a Lagrangian torus L in [58], where he proves that the class $[L]$ pairs trivially with any effective class, and a symplectic sphere with positive Chern number can be isotoped symplectically away from L.

In the case when L is a Lagrangian sphere in $S^2 \times S^2$ with a product symplectic form, Hind [23] constructed two transverse foliations of symplectic spheres where each sphere intersects L in a single point. This is used to show that every such L is Hamiltonian isotopic to the antidiagonal. For a Lagrangian sphere L in a symplectic Del Pezzo surface with Euler number at most 7, Evans showed in [15] that it can be displaced from certain symplectic spheres with positive Chern number up to Hamiltonian isotopy, and applied this displacement result to prove the uniqueness of Hamiltonian isotopy class of Lagrangian spheres.

In section 3, we generalize Evans' displacement result in two ways, the first being

Theorem 1.1. *Let L be a Lagrangian sphere in a symplectic 4-manifold (M, ω), and $A \in H_2(M; \mathbb{Z})$ with $A^2 \geq -1$. Suppose A is represented by a symplectic sphere C. Then C can be isotoped symplectically to another representative of A which intersects L minimally.*

In this paper all surfaces are smooth, embedded, connected, and oriented. We say that two closed surfaces intersect *minimally* if they intersect transversely at $|k|$ points where k is the homological intersection number.

The second generalization is for symplectic surfaces of arbitrary genus in manifolds with $b^+ = 1$. To state it let $\mathcal{E}_\omega$ be the set of $\omega-$exceptional classes:

$$\{E \in H_2(M, \mathbb{Z}) : E \text{ is represented by an } \omega\text{-symplectic } (-1) \text{ sphere}\}.$$

Theorem 1.2. *Suppose (M, ω) is a symplectic 4-manifold with $b^+ = 1$ and L is a Lagrangian sphere. Assume $A \in H_2(M, \mathbb{Z})$ satisfies $\omega(A) > 0, A^2 > 0$ and $A \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$. Then there exists a symplectic surface in the class nA intersecting L minimally for large $n \in \mathbb{N}$.*

One consequence of Theorem 1.2 is that we are able to effectively perform the Lagrangian-relative inflation procedure when $b^+ = 1$ (Section 5).

This turns out useful in dealing with a variety of questions, especially the existence of Lagrangian spheres. To approach this question, it is convenient to introduce the following definition.

Definition 1.3. A class ξ is called *K_ω-null spherical* if $\xi^2 = -2, K_\omega(\xi) = 0$ and it is represented by a smooth sphere. Here K_ω is the symplectic canonical class.

We classify K-null spherical classes in any (M, ω) with $\kappa = -\infty$. Recall that $\kappa(M, \omega)$ is the *Kodaira dimension* of (M, ω) (see for example [32]). κ

takes values in the set $\{-\infty, 0, 1, 2\}$, and $\kappa(M, \omega) = -\infty$ exactly when (M, ω) is symplectic rational or ruled. The classification of K_ω-null spherical classes, together with the Lagrangian-relative inflation, enables us to further show that the obvious necessary condition for the existence of a Lagrangian sphere in (M, ω) is also sufficient.

Theorem 1.4. *Let (M, ω) be a symplectic 4-manifold with $\kappa = -\infty$. $\xi \in H_2(M; \mathbb{Z})$ is represented by a Lagrangian sphere if and only if ξ is K_ω-null spherical and $\omega(\xi) = 0$.*

On the other hand, as in [15], Theorem 1.1 is useful in establishing uniqueness results for *rational manifolds*. A rational manifold is $\mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2$ or $S^2 \times S^2$. When M is a rational manifold (M, ω) is called a *symplectic rational manifold*. A symplectic rational manifold (M, ω) which is monotone, i.e. $[\omega] = K_\omega$, is also called a *symplectic Del Pezzo surface*.

Theorem 1.5. *Let (M, ω) be a symplectic rational manifold with Euler number $\chi \leq 7$, and ξ a K_ω-null spherical class with $\omega(\xi) = 0$. If ξ is not characteristic when $\chi = 6$, then Lagrangian spheres in ξ are unique up to Hamiltonian isotopy.*

This was due to Hind ([23]) in the case of $S^2 \times S^2$, and to Evans ([15]) for symplectic Del Pezzo surfaces with Euler number up to 7. Notice that this is equivalent to the transitivity of the Hamiltonian group action on the space of homologous Lagrangian spheres. The proof of Theorem 1.5 will be presented in Section 6. We believe that the uniqueness still holds when $\chi = 6$ and ξ is characteristic. However, the condition $\chi \leq 7$ in Theorem 1.5 is necessary, demonstrated by Seidel's twisted Lagrangian spheres in symplectic Del Pezzo surfaces with $\chi \geq 8$ ([51]).

Further, we prove:

Theorem 1.6. *Let (M, ω) be a symplectic rational manifold, and ξ a K_ω-null spherical class with $\omega(\xi) = 0$. If ξ is not characteristic when $\chi = 6$, then Lagrangian spheres in ξ are unique up to smooth isotopy.*

In the monotone case this was again due to Evans ([17]). We expect the extra condition being non-characteristic when $\chi = 6$ will eventually be removed. In fact, we are not aware of examples of homologous but not smoothly isotopic Lagrangian spheres in any symplectic 4-manifolds. For Lagrangian tori, such examples in a primitive homology class were first constructed by Vidussi in [57], and null-homologous ones were further constructed by Fintushel and Stern in [18].

We also conjecture the following version of uniqueness.

Conjecture 1.7. *For any two homologous Lagrangian spheres L_1 and L_2 in a symplectic rational manifold (M, ω), there exists $\phi \in Symp_h(M, \omega)$ such that $\phi(L_1) = L_2$.*

In other words, the Torelli part $Symp_h(M,\omega)$, which is the subgroup of $Symp(M,\omega)$ acting trivially on homology, should also act transitively on the space of Lagrangian spheres in a fixed homology class. Evans [16] calculated explicitly the homotopy type of $Symp_h(M,\omega)$ when (M,ω) is a symplectic Del Pezzo surface with $\chi \leq 8$ (also known to M.Pinnsonault). In particular, when $\chi \leq 7$, it is connected thus agreeing with $Ham(M,\omega)$. In our upcoming work [39] we will extend the connectedness to the non-monotone case.

It turns out that we are able to calculate the non-Torelli part of the symplectic mapping class group from Theorem 1.4. Recall that each Lagrangian sphere L gives rise to a symplectomorphism, well defined up to isotopy (see [51] and 2.1.1), which is denoted by τ_L and called the Lagrangian Dehn twist along L.

Theorem 1.8. *Let (M,ω) be a symplectic 4-manifold with $\kappa = -\infty$. Then the homological action of $Symp(M,\omega)$ is generated by Lagrangian Dehn twists. In other words, for any $f \in Symp(M,\omega)$, there are Lagrangian spheres L_i such that $f_* = (\tau_{L_1})_* \circ (\tau_{L_2})_* \circ \cdots \circ (\tau_{L_r})_*$.*

In the homological level, Theorem 1.8 could be viewed as a symplectic version of a classical theorem of M. Noether, which asserts that a birational automorphism of $\mathbb{C}P^2$ (also known as *plane Cremona map*) can be decomposed into a series of *ordinary quadratic transformations* (see [1] for a complete account).

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2 SFT of Lagrangian S^2

2.1 Geometry of T^*S^2

We first recall some standard facts of T^*S^2. Consider the embedding of the unit sphere in $\mathbb{R}^3$, which induces a symplectic embedding of T^*S^2 into $T^*\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$. In terms of the coordinates $(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3$, T^*S^2 is thus given by equations ([51], [15]):

$$\{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u| = 1, u \cdot v = 0)\}, \tag{2.1}$$

and the symplectic form is the restriction of $\omega_{can} = d\lambda_{can} = \sum dv_j du_j$ on $\mathbb{R}^6$, where the Liouville form $\lambda_{can} = \sum v_j du_j$ is also well-defined. (2.1) provides a Lagrangian splitting of the tangent bundle of T^*S^2 into the horizontal $u-$direction and the vertical $v-$direction.

Here is another useful model. Consider the affine quadric $Q = \{z_1^2+z_2^2+z_3^2 = 1\} \subset \mathbb{C}^3$. In terms of $u = \text{Re } z \in \mathbb{R}^3$ and $v = \text{Im } z \in \mathbb{R}^3$, Q is described by $|u|^2 - |v|^2 = 1, u \cdot v = 0$. Therefore $(u, v) \to (-\frac{u}{|u|}, v|u|)$ is a diffeomorphism from Q to T^*S^2. Moreover, if we restrict ω_{can} on $\mathbb{R}^6$ to Q, the diffeomorphism is in fact a symplectomorphism.

2.1.1 Symplectomorphisms of T^*S^2

The symplectomorphism group of T^*S^2 contains some compact subgroups. For each $l > 0$, denote $T_l^*S^2$ to be the open disk bundle with $|v| < l$, and H_l the sphere bundle of length l. The isometry group of S^2, $SO(3)$, acts on (T^*S^2, ω_{can}) as symplectomorphisms preserving each H_l.

The Hamiltonian function $Z(u, v) = \frac{1}{2}|v|^2$ generates a circle action on T^*S^2, agreeing with the cogeodesic flow. If we apply the symplectic cut operation in [31] to $\overline{T_l^*S^2}$ along H_l, we obtain $S^2 \times S^2$ with a monotone symplectic form (see for example [3]). In other words, $T_1^*S^2$ embeds into a monotone $S^2 \times S^2$ as the complement of the diagonal Δ.

The mapping class group of the compactly supported symplectomorphism group of (T^*S^2, ω_{can}) is non-trivial. In fact, it is the infinite cyclic group generated by a model Dehn twist of the zero section ([50]).

To define the model Dehn twist, consider the Hamiltonian function $T(u, v) = |v|$ on $T^*S^2 \backslash \{\text{zero section}\}$, whose Hamiltonian vector field is the unit field $(v/|v|, 0)$. The induced circle action is

$$\sigma_t(u, v) = (\cos(t)u + \sin(t)\frac{v}{|v|}, \cos(t)v - \sin(t)|v|u).$$

Notice that σ_π is the antipodal map $A(u, v) = (-u, -v)$, which extends smoothly over the zero section. Now choose a function $\rho : \mathbb{R} \to \mathbb{R}$ satisfying $\rho(t) = 0$ for $t \gg 0$ and $\rho(-t) = \rho(t) - t$. The Hamiltonian flow of $\rho(T)$ is $\sigma_{t\rho'(|v|)}(u, v)$. Since $\rho'(0) = 1/2$, the time 2π map extends smoothly over the zero section as the antipodal map. The resulting compactly supported symplectomorphism $\tau(u, v)$ of T^*S^2 is called a model Dehn twist.

There is a smooth isotopy with compact support from τ^2 to the identity, but no such symplectic isotopies exist.

2.1.2 Contact geometry of sphere bundles

The length l sphere bundle $H_l = \{|v| = l\}$ is a contact manifold with contact form λ_{can}. At the point (u, v) the contact plane distribution $\xi = \ker \lambda_{can}$ is

spanned by $(u \times v, 0)$ and $(0, u \times v)$.

The Reeb vector field at (u, v) is the vector field $(v, 0)$. Thus there are two dimensional simple Reeb orbits, all with the same period, and they foliate H_l. This is a special case of a Reeb flow of Morse-Bott type. In particular, the Reeb flow agrees with the cogeodesic flow of S^2 with round metric.

The vector fields $(u \times v, 0)$ and $(0, u \times v)$ provide a global trivialization Φ of ξ. With respect to Φ, the action of the Reeb flow on ξ along any Reeb orbit in H_l is considered as a path of matrices in $sp(2, \mathbb{R})$, whose Maslov index is defined to be the Conley-Zehnder index of the orbit ([13], [49]). From the calculation in [23] (see also [15]), simple Reeb orbits have Conley-Zehnder index 2.

H_l is in fact a contact-type hypersurface in $T^*_{l+\epsilon}S^2$, where the Liouville vector field is $(0, v)$. In particular, $\overline{T^*_l S^2} = \{|v| \leq l\}$ is a Liouville domain with convex boundary H_l.

2.1.3 Cylindrical coordinates

To apply SFT, we need to change to cylindrical coordinates. Consider a diffeomorphism $\Psi : T^*S^2 \to T^*S^2$, $(u, v) \to (u, \psi(|v|)v/|v|)$, where $\psi : [0, \infty) \to [0, \infty)$ is a smooth increasing function such that $\psi(s) = s$ for s small, and $\phi(s) = e^s$ for $s > r$. Ψ is the identity near the zero section, and $(T^*S^2, \Psi^*\omega_{can})$ is a symplectic manifold with one positive cylindrical end. Let $\omega = \Psi^*\omega_{can}$.

Then $(T^*_l S^2, \omega)$ is still a Liouville domain, with the Liouville field given by the unit field $\eta = (0, v/|v|)$ for $|v| > r$. Moreover, (T^*S^2, ω) is the (cylindrical) symplectic completion of $(T^*_l S^2, \omega)$.

On H_l, the contact form is $\lambda_l = \frac{\psi(l)}{l}\lambda$, and the Reeb vector field at (u, v) is $R_l = (\frac{l}{\psi(l)}v, 0)$.

2.2 Lagrangian S^2 and good almost complex structures

Let $L \subset (M, \omega)$ be a Lagrangian two sphere. From the Weinstein neighborhood theorem, the Lagrangian sphere L has a neighborhood U symplectomorphic to $(T^*_{2r}S^2, \omega_{can})$ for some small $r > 0$. Denote the symplectomorphism by Ξ. Let $U_l = \Xi^{-1}(\overline{T^*_l S^2})$ for $l < 2r$, and $W_l = M \backslash U_l$ be the complement of U_l.

In particular, $H = \partial U_l$ is a contact-type hypersurface with contact form $\lambda = \Xi^{-1}_* \lambda_l$.

2.2.1 J^0_t on T^*S^2

Following [23], we make a specific choice of $\omega-$compatible almost complex structure J^0 on T^*S^2 as follows: near the zero section, $J^0(X, 0) = (0, X)$; and for

$|v| > r$,

$$J^0|_{(u,v)}(v,0) = (0, \frac{\psi(l)}{l}v), \quad J^0|_{(u,v)}(u\times v, 0) = (0, u\times v).$$

J^0 is $SO(3)$-invariant, and J^0 is adjusted in the sense that, for $|v| > r$, it is $\frac{\partial}{\partial s}$−invariant, sending the Liouville field to the Reeb field.

Choose $l \in (r, 2r)$. When restricted to the Liouville domain $(\overline{T_l^* S^2}, \omega)$, $J^0|_{\overline{T_l^* S^2}}$ is adjusted in the collar neighborhood $r < |v| \le l$, and its cylindrical completion is canonically identified with (T^*S^2, J^0).

We need to further consider a deformation J^0_t of J^0. Let $V_t = [-t-\epsilon, t+\epsilon]$ and $\beta_t : V_t \to [-\epsilon, \epsilon]$ be a strictly increasing function with $\beta_t(s) = s + t$ on $[-t-\epsilon, -t-\epsilon/2]$ and $\beta_t(s) = s - t$ on $[t+\epsilon/2, t+\epsilon]$. Define a smooth embedding $f_t : V_t \times H_l \to T^*S^2$ by:

$$f_t(s,m) = (\beta_t(s) + l, m).$$

Let $\bar{J}_t$ be the $\frac{\partial}{\partial s}$−invariant almost complex structure on $V_t \times H_l$ such that $\bar{J}_t(\frac{\partial}{\partial s}) = R_l$ and $\bar{J}_t|_\xi = J^0|_\xi$. Glue the almost complex manifold $(T^*S^2 \backslash f_t(V_t \times H_l), J^0)$ to $(V_t \times H_l, \bar{J}_t)$ via f_t to obtain the family of almost complex structures J^0_t on T^*S^2.

Notice that each J^0_t agrees with J^0 away from the collar $l - \epsilon < |v| < l + \epsilon$. And on this collar, it agrees with J^0 on ξ, while $J^0_t|_{(u,v)}(v,0) = (0, \frac{d\beta_t^{-1}}{ds}|_{s=|v|-l}\frac{\psi(l)}{l}v)$.

On the other hand, via f_t, J^0_t restricted to $\overline{T_l^* S^2}$ is the same as J^0 on $\overline{T_{l+t}^* S^2}$. In particular, J^0_∞ can be viewed an almost complex structure on T^*S^2, which is in fact equal to J^0.

2.2.2 Neck-stretching on M

We say that an almost complex structure J on M is *adjusted* to $H = \partial U_l$ with respect to the Liouville vector field $\Xi_*^{-1}(\eta)$, if in a tubular neighborhood of H, J is invariant under the flow $\Xi_*^{-1}(\eta)$, $J(\Xi_*^{-1}(\eta))$ is the Reeb vector field on H, and J preserves the contact plane field ζ defined by the contact structure $i_\eta\omega$.

Following [17] consider the following Fréchet manifold of adjusted almost complex structures:

$$\overline{\mathcal{J}} = \{J \in \mathcal{J}_\omega : J = \Xi_*^{-1}J^0 \text{ on } U\}. \tag{2.2}$$

Given $J \in \overline{\mathcal{J}}$, define

$$J_t = J \text{ on } X\backslash U, \quad J_t = \Xi_*^{-1}J^0_t \text{ on } U.$$

Notice that J_t is in fact the *neck-stretching* of the adjusted J along ∂U_l with respect to $\Xi_*^{-1}(\eta)$. Fix a sequence $\{t_i \in \mathbb{R} : t_i \to +\infty\}$, we further define a sequence of Fréchet manifolds of adjusted almost complex structures:

$$\overline{\mathcal{J}}(i) = \{J \in \mathcal{J}_\omega : J = \Xi_*^{-1}J^0_{t_i} \text{ in } U\}.$$

From the explicit description of J_{t_i} in 2.2.1, we can reverse the neck-stretching, thus there is a diffeomorphism $P_i : \overline{\mathcal{J}}(i) \to \overline{\mathcal{J}}$.

When $i \to \infty$ the neck-stretching process results in an almost complex structure J_∞ on the union of symplectic completions $\overline{W}$ and $\overline{U}$ of W and U_r. $\overline{W}$ and $\overline{U}$ are two open symplectic manifolds with cylindrical ends, with $(\overline{U}, J^\infty)$ being (T^*S^2, J^0). J_∞ on the cylindrical end of $\overline{W}$ can be described explicitly: one simply extends η in the obvious way, and endows an η-adjusted almost complex structure which still restricts to J on ζ as above.

To describe the limits of pseudo-holomorphic curves under the deformation J_t, we need another open symplectic manifold. Let SH be the symplectization of the contact manifold H. We endow SH again the η-adjusted almost complex structure as on the cylindrical ends of $\overline{W}$ and $\overline{U}$, and also denote it by J_∞.

2.3 Finite energy holomorphic curves

Suppose S is a closed Riemann surface and $\Gamma \subset S$ an ordered finite set of punctures.

Let (Z, ω) be any of the three symplectic 4-manifolds $\overline{W}$, $\overline{U}$, or SH, each equipped with the adjusted almost complex structure J_∞. Denote E^+ (E^-) to be the positive (negative) end, which is allowed to be empty.

Notice that, since $J^0_\infty(\frac{\partial}{\partial s}) = R_l$, and ξ is $J^0_\infty-$invariant, the real trivialization Φ of ξ on H_l canonically induces a complex trivialization of the complex rank 2 bundle (TZ, J_∞) along $E^\pm$, which we still denote by Φ.

Suppose $u : S\backslash\Gamma \to Z$ is a proper map. u is called simple if it does not factor through a multiple cover.

Let $u^\pm$ be the restriction to $u^{-1}(E^\pm)$. Then $u^\pm$ has the form $(a_\pm, v_\pm)$ in coordinates $\mathbb{R}_\pm \times H$. Consider the set $\mathcal{C}$ of functions $\phi_\pm : \mathbb{R}_\pm \to \mathbb{R}$ with integral 1.

The $\lambda-$energy of a map $u : S\backslash\Gamma \to Z$ is defined by

$$E_\lambda(u) = \sup_{\phi_\pm \in \mathcal{C}} \Big(\int_{u^{-1}(E^+)} (\phi_+ \circ a_+) da_+ \wedge v_+^* \lambda + \int_{u^{-1}(E^-)} (\phi_- \circ a_-) da_- \wedge v_-^* \lambda \Big).$$

The energy of u is then given by

$$E(u) = \int_{u^{-1}(Z\backslash(E^+ \cup E^-))} u^*\omega + E_\lambda(u).$$

u is called a finite energy map if $E(u) < \infty$. Since we are in the Morse-Bott situation, i.e the Reeb flow on $E^\pm$ is Morse-Bott, finite energy $J_\infty-$holomorphic curves are asymptotic to periodic orbits in $E^\pm$ ([11]).

Suppose S has genus g, and u has s^+ positive punctures converging to $\gamma_k^+, 1 \le k \le s^+$, s^- negative punctures converging to $\gamma_k^-, 1 \le k \le s^-$. Two

such maps u and u' are called equivalent if there is a biholomorphism $h : (S, \Gamma) \to (S', \Gamma')$ such that $u = u' \circ h$.

Each u is associated with a CR operator, and u is called (SFT) regular if the operator is surjective ([15]). Denote the index of this operator by index(u). To state the index formula, suppose $n_i^+ = \mathrm{cov}(\gamma_i^+)$ and $n_j^- = \mathrm{cov}(\gamma_j^-)$, where $\mathrm{cov}(\gamma)$ denotes the multiplicity of γ over a simple Reeb orbit. Since each Reeb orbit is in a 2 dimensional manifold and has CZ index 2, following the computation on [23] and [11], we have:

$$\mathrm{index}(u) = -(2-2g) + 2s + 2c_1^{\Phi}([u]) + \sum_{k=1}^{s^+} 2\mathrm{cov}(\gamma_k^+) - \sum_{k=1}^{s^-} 2\mathrm{cov}(\gamma_k^-). \quad (2.3)$$

Here $c_1^{\Phi}(TZ)$ is the relative first Chern class of (TZ, J_∞) relative to the trivialization Φ along the ends, $[u]$ is the relative homology class of u ([15]).

The following is a very special case of a theorem due to Wendl, which states that for certain u, the SFT regularity is automatic.

Theorem 2.1 (Wendl, [59]). *Suppose (W, J) is a 4-dimensional almost complex manifold with cylindrical end modelled on contact manifolds foliated by Morse-Bott Reeb orbits, and $u : (S, \Gamma) \to W$ is a embedded pseudo-holomorphic curve with punctures. If*

$$\mathrm{index}(u) > 2g + 2|\Gamma| - 2, \quad (2.4)$$

then u is regular.

2.3.1 Regular holomorphic curves in $\overline{W}$

We discuss the SFT transversality in $\overline{W}$.

Remark 2.2. It is well-known, for example by Remark 3.2.3 in [45] that, to achieve transversality for the moduli space of pseudo-holomorphic curves, it suffices to consider the space of $\omega-$compatible almost complex structure which is fixed on an open set, provided that every pseudo-holomorphic curve representing the class passes through its complement.

Recall that a Baire set is the countable intersection of open and dense sets. Since no punctured pseudo-holomorphic curves can lie completely inside $\overline{U}$, the arguments to prove Theorem 5.22 in [15] also proves:

Proposition 2.3. *Using notations in Section 2.2, there exists a Baire set in $\bar{\mathcal{J}}_W \subset \bar{\mathcal{J}}$ such that for any $J \in \bar{\mathcal{J}}_W$, J_∞ is SFT regular in the sense that every finite energy $J_\infty-$holomorphic curve u is regular.*

We will need variations of other standard transversality results about pseudo-holomorphic curves, where the above observation will be crucial.

2.3.2 Genus 0 curves in SH with a single simple asymptote

In SH we will encounter curves as in the following lemma.

Lemma 2.4. *Suppose $u : C \to SH$ is a $J_\infty-$holomorphic curve of genus 0 in SH with one positive end asymptotic to a simple Reeb orbit. Then u is a trivial cylinder.*

Proof. The proof is contained in Lemma 7.5 [15] (see also [23], [12]). We briefly recall the main points. Since each Reeb orbit is non-trivial in $\pi_1(H)$ and C has genus 0, there has to be at least one negative puncture. On the other hand, since $E_\lambda(u) \geq 0$ and all Reeb orbits have the same period, u has at most one negative puncture, which has to be simple. Thus u is a trivial cylinder. □

2.3.3 J^0-holomorphic planes in T^*S^2

In T^*S^2 we need to consider embedded holomorphic planes with one (positive) end asymptotic to a simple Reeb orbit.

As mentioned, on T^*S^2, J_∞ is the same as J^0. Notice that J^0 interchanges the two summands of the Lagrangian splitting of the tangent bundle of T^*S^2. Thus $\det(TT^*S^2, J^0)$ is canonically trivialized since the Lagrangian horizontal two plane bundle is orientable. The expected dimension of the moduli space of embedded J^0-holomorphic plane u with one (positive) end asymptotic to a simple Reeb orbit is thus given by

$$\text{index}(u) = -2 + 2 + 2 = 2. \tag{2.5}$$

This follows from the general index formula (2.3), and the vanishing of c_1^Φ for all punctured curves in T^*S^2.

It is proved in Lemmas 8 and 9 and Section 4 in [23] that if $\tilde{J}^0$ is close to J^0 and any embedded $\tilde{J}^0-$holomorphic planes with one simple puncture is regular, then $\tilde{J}^0$ enjoys the following properties:

(1) There are two $\tilde{J}^0$-foliations $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ in T^*S^2, such that there is a one-one correspondence from simple Reeb orbits to planes in each foliation;

(2) Each element in $\mathcal{F}_\alpha$ ($\mathcal{F}_\beta$, resp.) intersects the zero-section at a single point positively (negatively, resp.).

We will call the planes in $\mathcal{F}_\alpha$ ($\mathcal{F}_\beta$, resp.) α-planes (β-planes, resp.).

One consequence of (2.5) is that we can appeal to Wendl's Theorem 2.1 to conclude that each embedded J^0-holomorphic planes with one simple puncture is regular. In particular, J^0 also satisfies the above properties. Furthermore, we have

Lemma 2.5. *A J^0-holomorphic plane in T^*S^2 asymptotic to a simple Reeb orbit belongs to either $\mathcal{F}_\alpha$ or $\mathcal{F}_\beta$. Moreover, an α-plane and a β-plane intersect transversally if they do not share the same asymptote.*

Proof. The proof is largely similar to Lemma 8 in [23]. One could think of T^*S^2 topologically as a neighborhood of $\bar{\Delta}$, the anti-diagonal in $S^2 \times S^2$. The complement is then a disk bundle over Δ the diagonal, of which the boundary of disk fibers coincides with the simple Reeb orbits in T^*S^2. One can then glue these disks to elements in $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$, resulting in two foliations in $S^2 \times S^2$, with classes $[S^2 \times pt]$ and $[pt \times S^2]$, respectively. Suppose we have a J^0-holomorphic plane P in U asymptotic to some simple Reeb orbit γ, which does not belong to either $\mathcal{F}_\alpha$ nor $\mathcal{F}_\beta$, it must intersect some $P_\alpha \in \mathcal{F}_\alpha$ and $P_\beta \in \mathcal{F}_\beta$ positively, where P_α and P_β have asymptotes $\gamma_\alpha, \gamma_\beta$ which are different from γ. Now P, P_α and P_β can all be capped in $S^2 \times S^2$ by the above procedure, resulting in three spheres intersecting only in U. By construction, the sphere formed by capping P has positively intersection with both $[S^2 \times pt]$ and $[pt \times S^2]$, but intersects Δ at a single point, which leads to a contradiction.

The second assertion can be proved similarly, for if $\gamma_\alpha \neq \gamma_\beta$, the capped sphere does not have intersection in the complement of T^*S^2, so they must intersect inside T^*S^2 for homological reason.

□

Remark 2.6. If we do not appeal to Wendl's automatic transversality result, instead of J^0, we could simply use a fixed $\tilde{J}^0$ satisfying the properties above throughout the paper.

2.3.4 SFT compactness

Following [23] we briefly recall the relevant compactness results in the symplectic field theory adapted to our case. For detailed expositions on the subject, we refer the readers to [12] and [11].

Let $M_\infty = \overline{W} \cup SH \cup \overline{U}$, and J_∞ be the almost complex structure defined as in section 2.2. Let Σ be a Riemann surface with nodes. A *level-k holomorphic building* consists of the following data:

(i) (level) A labelling of the components of $\Sigma \backslash \{\text{nodes}\}$ by integers $\{1, \cdots, k\}$ which are the *levels*. Two components sharing a node differ at most by 1 in levels. Let Σ_r be the union of the components of $\Sigma \backslash \{\text{nodes}\}$ with label r.

(ii) (asymptotic matching) Finite energy holomorphic curves $v_1 : \Sigma_1 \to U$, $v_r : \Sigma_r \to SH$, $2 \leq r \leq k-1$, $v_k : \Sigma_k \to W$. Any node shared by Σ_l and Σ_{l+1} for $1 \leq l \leq k-1$ is a positive puncture for v_l and a negative

puncture for v_{l+1} asymptotic to the same Reeb orbit γ. v_l should also extend continuously across each node within Σ_l.

Now for a given stretching family $\{J_{t_i}\}$ as previously described, as well as J_{t_i}-curves $u_i : S \to (M, J_{t_i})$, we define the Gromov-Hofer convergence as follows:

A sequence of J_{t_i}-curves $u_i : S \to (M, J_{t_i})$ is said to be *convergent to a level-k holomorphic building* v in Gromov-Hofer's sense, using the above notations, if there is a sequence of maps $\phi_i : S \to \Sigma$, and for each i, there is a sequence of $k-2$ real numbers t_i^r, $r = 2, \cdots, k-1$, such that:

(i) (domain) ϕ_i are locally biholomorphic except that they may collapse circles in S to nodes of Σ,

(ii) (map) the sequences $u_i \circ \phi_i^{-1} : \Sigma_1 \to U$, $u_i \circ \phi_i^{-1} + t_i^r : \Sigma_r \to SH$, $2 \leq r \leq k-1$, and $u_i \circ \phi_i^{-1} : \Sigma_k \to W$ converge in C^∞-topology to corresponding maps v_r on compact sets of Σ_r.

Now the celebrated compactness result in SFT reads:

Theorem 2.7 ([12]). *If u_i has a fixed homology class, there is a subsequence t_{i_m} of t_i such that $u_{t_{i_m}}$ converges to a level-k holomorphic building in the Gromov-Hofer's sense.*

3 Minimal intersection

In this section we prove Theorems 1.1 and 1.2. There are two main ingredients, the symplectic Seiberg-Witten theory which produces embedded, connected pseudo-holomorphic submanifolds for a class of compatible almost complex structures suitable for applying symplectic field theory. Via neck stretching the symplectic field theory then produces in the limit the desired symplectic surfaces which intersect L minimally.

3.1 Embedded and nodal pseudo-holomorphic submanifolds

We first introduce some notations. All surfaces in this section are closed. Given a class $e \in H_2(M, \mathbb{Z})$, let $\eta_\omega(e)$ be the $\omega-$symplectic genus of e:

$$\eta_\omega(e) = \frac{e \cdot e + K_\omega(e) + 2}{2}. \tag{3.1}$$

This is exactly the genus of a connected embedded ω-symplectic surface in class e (if there is one) from the adjunction formula.

Also define the dimension of e

$$d(e) = \frac{-K_\omega(e) + e \cdot e}{2}. \tag{3.2}$$

$d(e)$ is the expected dimension of the moduli space of embedded pseudo-holomorphic curve of genus $\eta_\omega(e)$ in the class e. In terms of $\eta_\omega(e)$, $d(e)$ can also be expressed as:

$$d(e) = -K_\omega(e) + \eta_\omega(e) - 1.$$

Suppose C is a compact, connected, pseudo-holomorphic submanifold of M. Then C has the structure of a Riemann surface and it represents a nonzero class $[C]$. Moreover, there is a canonically associated elliptic operator

$$D_C : \Gamma(N) \to \Gamma(N \otimes T^{1,0}C), \tag{3.3}$$

where N is the normal bundle of C. D_C is called the normal operator of C and the index of D_C is exactly given by $d([C])$.

Fix a set Ω of $d([C])$ distinct points. If $\Omega \subset C$, then we can define the operator

$$D_C \oplus ev_\Omega : \Gamma(N) \to \Gamma(N \otimes T^{1,0}C) \oplus (\oplus_{p\in\Omega} N|_p).$$

The index of $D_C \oplus ev_\Omega$ is 0. And the kernel of $D_C \oplus ev_\Omega$ should be thought of as giving a sort of Zariski tangent space to the space of pseudo-holomorphic embeddings of C in M containing the subset Ω (as a point in the space of smooth embeddings). C is called (J, Ω) non-degenerate if the operator $D_C \oplus ev_\Omega$ has trivial cokernel (and also trivial kernel).

D_C is a real CR operator on (C, N). For such operators, there is the following automatic transversality result.

Theorem 3.1 ([26], [27]). *Let Σ be a Riemann surface of genus g, and let L be a complex line bundle over Σ. Let D be a real CR operator. Suppose $c_1(L) \geq 2g - 1$, then* $\mathrm{cokerD} = 0$.

We will show in the next two subsections that in two situations, given a class e, there is a Baire set of pairs (J, Ω) for which there are **connected** $J-$holomorphic submanifolds of genus $\eta_\omega(e)$ through Ω. The Baire property is shown by first setting up universal models of various type of pseudo-holomorphic curves, and then exploiting the Fredoholm properties of D in conjunction with the Sard-Smale theorem and the Gromov compactness theorem to rule out unwanted behavior for generic pairs (J, Ω).

We also need to generalize to the case of a nodal pseudo-holomorphic submanifold in the sense of Sikorav ([53]). Let $\Sigma = \cup \Sigma_i$ be a nodal Riemann surface, where Σ_i are the irreducible components. A $J-$holomorphic map $f : \Sigma \to (M, J)$ is said to be nodal if f has distinct tangents along two branches at each node. For our purpose, we call a nodal curve f a nodal submanifold if f

is an embedding on each Σ_i. Thus a nodal submanifold is a union of embedded submanifolds intersecting transversally. Let $C_i = f(\Sigma_i)$.

For a nodal submanifold, the analogue of (3.3), $D_{\cup C_i}$, is defined in Section 4 in [53] in terms of the normalization of Σ. $D_{\cup C_i}$ is elliptic and its index is simply given by $\sum_i d([C_i])$.

In this case, for each i, fix a subset $\Omega_i \subset C_i$ with $d([C_i])$ distinct points and not containing any of the nodes. Then the operator $D_{\cup C_i} \oplus ev_{\cup \Omega_i}$ is an elliptic operator with index zero, and f is called non-degenerate if $D_{\cup C_i} \oplus ev_{\cup \Omega_i}$ has trivial cokernel.

The automatic transversality in this context, Corollary 2 in [53], implies that $D_{\cup C_i} \oplus ev_{\cup \Omega_i}$ is onto if

$$-K_\omega([C_i]) > 0, \quad \text{for each } i. \tag{3.4}$$

3.1.1 Symplectic spheres

Suppose C is an embedded symplectic sphere with self-intersection at least -1. In this case

$$d([C]) = -K_\omega([C]) - 1, \quad [C] \cdot [C] = -K_\omega([C]) - 2. \tag{3.5}$$

The following should be well known. We present some details in view of the generalization to certain configurations, Proposition 3.4.

Proposition 3.2. *Let (M, ω) be a symplectic 4-manifold, $e \in H_2(M; \mathbb{Z})$ with $e^2 \geq -1$ a class represented by an embedded symplectic sphere C. Then there is a path connected Baire subset $\mathcal{T}_e$ of $\mathcal{J}_\omega \times M_{d(e)}$ such that a pair (J, Ω) lies in $\mathcal{T}_e$ if and only if there is a unique embedded $J-$holomorphic sphere in the class e containing Ω. Here M_d is the space of $d-$trples of distinct (but unlabeled) points in M. Consequently, any symplectic sphere in the class e is isotopic to C.*

Proof. Pick an almost complex structure $J \in \mathcal{J}_\omega$ such that C is J-holomorphic and $\Omega \subset C$.

Following [4] (Lemma 4 and formula (15)) and [27], let $P = -\sum_{z_i \in \Omega} z_i$ be the divisor of C and $\tilde{N} = N \otimes P$. Then there exists a real CR operator on $(C, \tilde{N})$,

$$\tilde{D}_C : \Gamma(\tilde{N}) \to \Gamma(\tilde{N} \otimes T^{1,0}C),$$

with the property that $\text{coker}\tilde{D}_C \cong \text{coker}(D_C \oplus ev_\Omega)$. Notice that, by (3.5),

$$c_1(\tilde{N}) = c_1(N) - d([C]) = e \cdot e - d(e) = -1.$$

From Theorem 3.1, $\tilde{D}$ is surjective.

Notice that $d(e) \geq 0$. Moreover, from the positivity of intersections and the fact that $e \cdot e = d(e) - 1$, C is the only connected J-sphere in e containing Ω. Since $\tilde{D}$ is surjective, C is regular with respect to (J, Ω). Thus we conclude that the genus 0 Gromov-Witten invariant of e passing through $d(e)$ points is ± 1, in particular, nonzero.

A marked $\mathbb{P}^1$ is a pair $(\mathbb{P}^1, \{z_i\})$ where $\{z_i\}$ is a set of unordered, distinct points. Now introduce the universal genus zero moduli space $\mathcal{P}$ associated to e, which is the space of $J-$holomorphic embedding $u : (\mathbb{P}^1, \{z_i\}_{i=1}^{d(e)}) \to (M, J)$ with $[u] = e$ for some $J \in \mathcal{J}_\omega$, modulo the automorphism of $\mathbb{P}^1$. $\mathcal{P}$ is a Frechet manifold ([45]). Moreover, the natural map π to $\mathcal{J}_\omega \times M_{d(e)}$, $(u, J, \{z_i\}) \to (J, \{u(z_i)\})$ is Fredholm. The argument above simply means that π is an isomorphism onto its image.

Similarly, for each possible singular type c, introduce the auxiliary universal moduli space $\mathcal{P}_c$. Each $\mathcal{P}_c$ is again a Frechet manifold and the projection $\pi_c : \mathcal{P}_c \to \mathcal{J}_\omega \times M_{d(e)}$ is Fredholm ([45]) with index at most -2. Notice that the image of π and the union of the images of π_c cover $\mathcal{J}_\omega \times M_{d(e)}$ by the non-triviality of the Gromov-Witten invariant. Since each π_c has negative index, the complement of the image of π_c is exactly the set of regular values of π_c, hence is Baire. This implies the image of π is Baire.

Now we show that the image of π is path connected. Let (J', Ω') be in the image of π. The Sard-Smale theorem implies that along a generic path (J_t, Ω_t) connecting (J, Ω) and (J', Ω'), for each t, (J_t, Ω_t) is either a regular value of projections π and π_c, or it is a singular value for one of the projections but the cokernel has dimension 1. Since each π_c has index -2 and π has no singular values, each (J_t, Ω_t) lies in $\mathcal{T}$.

Finally, notice that the path connected set $\mathcal{T}$ maps onto the space of symplectic spheres in the class e.

$\square$

For our application we need to take one step forward.

Definition 3.3. We call an ordered configuration of symplectic spheres $\cup C_i$ a *stable spherical symplectic configuration* if

1. $[C_i] \cdot [C_i] \geq -1$ for each i,
2. for any pair i, j with $i \neq j$, $[C_i] \neq [C_j]$, and $[C_i] \cdot [C_j] = 0$ or 1.
3. they are simultaneously $J-$holomorphic for some $J \in \mathcal{J}$.

The homological type refers to the set of homology classes $[C_i]$.

Notice that, by local positivity of intersection, 2 and 3 imply that C_i and C_j are either disjoint or intersect transversally at one point. In particular, it is a $J-$*nodal* submanifold. Further, since $C_i \cdot C_i \geq -1$, the condition (3.4) is satisfied by (3.5).

If we follow the arguments above, replacing Theorem 3.1 by Corollary 2 in [53], we obtain:

Proposition 3.4. *Suppose there is a stable spherical symplectic configuration $\cup_i C_i$ with type D. Then there is a path connected Baire subset $\mathcal{T}_D$ of $\mathcal{J}_\omega \times \prod_i M_{d([C_i])}$ such that a pair (J, Ω_i) lies in $\mathcal{T}_D$ if and only if there is a unique embedded $J-$holomorphic $D-$configuration with the $i-$th component containing Ω_i. Consequently, stable spherical symplectic configurations with the same homological type are isotopic.*

3.1.2 Gromov-Taubes invariants when $b^+ = 1$

Given a class e and a pair (J, Ω) in $\mathcal{J}_\omega \times M^{d(e)}$, introduce the set $\mathcal{H} \equiv \mathcal{H}(e, J, \Omega)$ whose elements are the unordered sets of pairs $\{(C_k, m_k)\}$ of disjoint, connected, $J-$holomorphic submanifold $C_k \subset M$ and positive integer m_k, which are constrained as follows:

1. If e_k is the fundamental class of C_k then $d_k \equiv d(e_k) \geq 0$.
2. If $d_k > 0$, then C_k contains a subset $\Omega_k \subset \Omega$ consisting of precisely d_k points.
3. The integer $m_k = 1$ unless C_k is a torus with trivial normal bundle.
4. $\sum_k m_k e_k = e$.

Notice that (3.2) and (3.1) imply that

- the only negative square components are spheres with square -1;
- a square 0 component is either a sphere or a torus;

To define the Gromov-Taubes invariant of a class e, a notion of admissibility of pair is introduced in [54]. The Gromov-Taubes invariant $GT(e)$ of e is then a suitably weighted count of $\mathcal{H}(e, J, \Omega)$ for an admissible (J, Ω), which is delicate at the presence of a toridal component with multiplicity higher than 1. When $b^+ = 1$, we will see that there are simple homological conditions to avoid such components.

It is rather involved to fully describe the precise meaning of admissible pairs, especially at the presence of a toridal component with multiplicity higher than 1. In fact, in the case $d(e) = 0$, Ω is the empty set, we are simply talking about the admissibility of J alone. Furthermore, if there are no toridal components, J is admissible if $\mathcal{H}(e, J)$ is a finite set, and each submanifold in a member of $\mathcal{H}(e, J)$ is non-degenerate.

It is also shown in [54] that the set of admissible pair is Baire. The argument is similar to the one in Proposition 3.2. In fact, by Remark 2.2, the intersection with each $\overline{\mathcal{J}}(i)$ is still Baire in $\overline{\mathcal{J}}(i)$ since U contains no closed pseudo-holomorphic curve.

When C is a symplectic sphere with self-intersection at least -1, it is easy to show that $GT([C]) = 1$ using arguments in Proposition 3.2. In general, when $b^+ = 1$, due to Taubes' SW$\Rightarrow$GT [55] and the Seiberg-Witten wall crossing formula, there are plenty of classes with non-trival GT invariant, and most of them are represented by connected embedded symplectic surfaces ([36], see also [7], [42], [34]) :

Proposition 3.5. *Let (M, ω) be a symplectic 4-manifold with $b^+ = 1$ and canonical class K_ω. Let $A \in H_2(M; \mathbb{Z})$ be a class satisfying the following properties:*

- *$A^2 > 0$ and $\omega(A) > 0$;*
- *$A - PD(K_\omega)$ is ω-positive and has non-negative square;*
- *$A \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$.*

Then A has non-vanishing GT invariant and A is represented by a connected embedded symplectic surface.

Lemma 3.6. *Let (M, ω) be a symplectic 4-manifold with $b^+ = 1$. Suppose $e \in H_2(M; \mathbb{Z})$ is a class with $\eta_\omega(e) \geq 2$, $e \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$, and $GT(e) \neq 0$. Then for any admissible (J, Ω), A has a connected $J-$holomorphic representative of genus $\eta_\omega(e)$.*

Proof. Suppose (J, Ω) is admissible. Let C be a $J-$holomorphic submanifold contributing to $GT(e)$. The condition that $e \cdot E \geq 0$ for all $E \in \mathcal{E}_\omega$ ensures C has no negative-square components. Since $b^+(M) = 1$, if C is disconnected, then all the components are homologous and have square 0. Thus C is either a union of spheres with square 0, or a union of tori with square 0. However, this contradicts the assumption that $\eta_\omega(e) \geq 2$ from the adjunction formula. Therefore C is a connected genus $\eta_\omega(e)$ surface as claimed.

$\square$

Furthermore, assume that $d(e) \geq 1$. Let $\{U_i\}_{i=1}^{d(e)}$ be a sequence of pairwisely disjoint Darboux chart. We consider the class of almost complex structures $\mathcal{F}_{\{U_i\}} \subset \mathcal{J}_\omega$ which is fixed and integrable on U_i. By Remark 2.2, there is an admissible pair $(\tilde{J}, \tilde{\Omega} = \{x_i\})$ with $\tilde{J} \in \mathcal{F}_{\{U_i\}}$ and $x_i \in U_i$. In particular, there is a connected embedded $\tilde{J}-$holomorphic curve $\tilde{C}$ through $\{x_i\}$ with $[\tilde{C}] = e$.

For any such $\tilde{J} \in \mathcal{F}_{\{U_i\}}$, let $p : (M', \tilde{J}_{\{x_i\}}) \to (M, J)$ be the complex blow-up of $(M, \tilde{J})$ at x_i. Denote each exceptional sphere by C_{x_i} and its neighborhood corresponding to U_i by U_i'. One can then endow M' with a symplectic form ω' compatible with $J_{\{x_i\}}$. (see Lemma 7.15 in [44]). Denote also $\mathcal{F}'_{\{U_i\}} \subset \mathcal{J}_{\omega'}$ to be the corresponding set of almost complex structures.

Lemma 3.7. *Given the same assumption in Lemma 3.6 and consider $(M', J_{\{x_i\}})$ as above. Let $E_i = [C_{x_i}]$, $i = 1, \cdots, d(e)$ and $A' = A - \sum_{1 \leq i \leq d(e)} E_i$. Then*

$GT_{\omega'}(A') \neq 0$, and for J in a Baire subset of $\mathcal{F}_{\{U_i\}}$, A' is represented by a connected $J_{\{x_i\}}-$holomorphic surface of genus $\eta_\omega(A)$, intersecting each C_{x_i} transversally at one point.

Proof. Now $-K_{\omega'}(A') = -K_\omega(A) - d(e)$ and $\eta_{\omega'}(A') = \eta_\omega(A)$. Since $d(A') = 0$, from the blow-up formula, Corollary 4.4 in [37], A' also has nontrivial GT invariant.

Since the only $J'-$holomorphic curves contained in $\cup U_i'$ are C_{x_i}, by Remark 2.2, the intersection of admissible almost complex structures on (M', ω') with $\mathcal{F}'_{\{U_i\}}$ is a Baire set in $\mathcal{F}'_{\{U_i\}}$.

To check the generic connectedness, by Lemma 3.6 we only need to verify the homological condition $A' \cdot E \geq 0$ for any $E \in \mathcal{E}_{\omega'}$. But as is shown above, there is a connected embedded $\tilde{J}-$holomorphic $\tilde{C} \subset M$, thus its proper transformation $\tilde{C}' \subset M'$ is $J_{\{x_i\}}$-holomorphic with $[\tilde{C}'] = A'$. Notice also that every E has a $J_{\{x_i\}}-$holomorphic representative since exceptional classes always have nontrivial GW invariant. Since the genus of $\tilde{C}'$ is positive, it is different from any component of E. By positivity of intersections, we have $[\tilde{C}'] \cdot E \geq 0$.

Finally, since $\mathcal{F}'_{\{U_i\}}$ and $\mathcal{F}_{\{U_i\}}$ are canonically identified via complex blowing up the x_i and complex blowing down the C_{x_i}, we obtain the required Baire subset of $\mathcal{F}_{\{U_i\}}$.

□

3.2 Proof of Theorems 1.1 and 1.2

We are ready to prove Theorem 1.1 and 1.2. For the convenience of exposition, we first investigate the behavior of generic J-holomorphic representatives in class A in neck-stretching, when the class A satisfies

$$-K_\omega(A) = 1 - \eta_\omega(A). \tag{3.6}$$

Firstly, regarding the fixed class A, we claim that there is a Baire set $\mathcal{J}_{reg}(A) \subset \overline{\mathcal{J}}$ such that for each $J \in \mathcal{J}_{reg}$, J_{t_i} is GT admissible for each i, and J_∞ is regular in the sense of SFT for $\overline{W}$. By Proposition 2.3 there is a Baire subset $\mathcal{J}'_{reg} \subset \overline{\mathcal{J}}$, such that for $J \in \mathcal{J}'_{reg}$, J_∞ is SFT regular. Recall from 2.2.2, $\overline{\mathcal{J}}(i) = \{J | J = J^0_{t_i} \text{ in } U\}$ and P_i is the identification of $\overline{\mathcal{J}}(i)$ with $\overline{\mathcal{J}}$. We have mentioned that, as all closed pseudo-holomorphic curves have to pass through $M \backslash U$, there is a Baire subset $\overline{\mathcal{J}}(i)' \subset \overline{\mathcal{J}}(i)$ such that each member is GT admissible. One then takes $\mathcal{J}_{reg}(A) = \cap_n P_n(\overline{\mathcal{J}}'_n) \cap \mathcal{J}'_{reg}$.

Fix $J \in \mathcal{J}_{reg}(A)$. By Lemma 3.6 there is a sequence of connected embedded J_{t_i}-holomorphic submanifolds C_{t_i}. If C_{t_i} does not intersect L for some $i < \infty$, the theorem follows. Now we assume that each C_{t_i} intersects L. This assumption will eventually lead to a contradiction when $[L] \cdot [C] = 0$ and is automatically satisfied if $[L] \cdot [C] \neq 0$.

By Theorem 2.7, there is a $k-$leveled curve C_∞ as a Gromov-Hofer limit of $\{C_{t_i}\}_{i=0}^\infty$: the piece in $M\backslash U_l$, which we call C_W or the W-part; the piece in the symplectization of $\partial U_l = \mathbb{R}P^3$ consisting of $k-2$ levels, which we call C_{SH} or the SH-part; the piece in U_l, which we call C_U or the U-part. Let us first examine the W-part.

Lemma 3.8. *Suppose* (3.6) *is satisfied. Then C_W is a, possibly unbranched covering, irreducible genus-$\eta_\omega(A)$ curve, and all asymptotic Reeb orbits are simple. Moreover, let $\bar{C}_W$ be the underlying simple curve, then the limits of punctures of $\bar{C}_W$ are pairwisely distinct.*

Proof. By the maximum principle, C_W is non-empty. Let $u_i : B_i \to W, 1 \leq i \leq q$, be the irreducible components of C_W and g_i the genus of B_i. Suppose u_i is a degree m_i multiple cover of $\bar{u}_i : \bar{B}_i \to W$.

Notice that

$$c_1^\Phi = 0 \text{ in } U \text{ and } S$$

implies that

$$\sum_{1\leq j\leq q} c_1^\Phi(TW)([u_j]) = -K_\omega(A). \tag{3.7}$$

From the description of Gromov-Hofer convergence in 2.3.4, we clearly have $\sum_{1\leq j\leq q} g_j \leq \eta_\omega(A)$. (3.6) then implies that

$$\sum_{1\leq j\leq q} c_1^\Phi(TW)([u_j]) \leq 1 - \sum_{1\leq j\leq q} g_j.$$

If $q > 1$, there must be some component, say B_1, with

$$c_1^\Phi(TW)([u_1]) \leq -g_1. \tag{3.8}$$

By (2.3), we have

$$\text{index}(\bar{u}_1) = -(2 - 2g(\bar{B}_1)) + 2\bar{s}_1^- + 2c_1^\Phi(TW)([\bar{u}_1]) - \sum_{k=1}^{\bar{s}_1^-} 2\text{cov}(\bar{\gamma}_k) \tag{3.9}$$

Here $\bar{s}_1^-$ is the total number of punctures of $\bar{u}_1$ and the $\bar{\gamma}_k$ are the asymptotic Reeb orbits. By our choice of J, $\text{index}(\bar{u}_1) \geq 0$, thus we must have

$$c_1^\Phi(TW)([\bar{u}_1]) \geq 1 - g(\bar{B}_1). \tag{3.10}$$

Notice that $c_1^\Phi(TW)([u_1]) = m_1 c_1^\Phi(TW)([\bar{u}_1])$. Since $2\bar{s}_1^- - \sum_{k=1}^{\bar{s}_1^-} 2\text{cov}(\bar{\gamma}_k) \leq 0$, by (3.8) we have

$$g_1 \leq m_1(g(\bar{B}_1) - 1).$$

But this is impossible by the Riemann-Hurwitz formula

$$(g_1 - 1) \geq m_1(g(\bar{B}_1) - 1). \tag{3.11}$$

This contradiction shows that C_W is irreducible, namely, given solely by u_1, when $J \in \mathcal{J}_{reg}$. By (3.10) and (3.7), we have

$$1 - \eta_\omega(A) = m_1 c_1^\Phi(TW)([\bar{u}_1]) \geq m_1(1 - g(\bar{B}_1)).$$

Since $g_1 \leq \eta_\omega(A)$, we have by (3.11), that

$$\eta_\omega(A) = g_1.$$

Notice that this also means u_1 is an unbranched covering. Now return to (3.10), we find that

$$(\bar{u}_1) = 2\bar{s}_1^- - \sum_{k=1}^{\bar{s}_1^-} 2\mathrm{cov}(\bar{\gamma}_k) \geq 0. \tag{3.12}$$

Hence we conclude that each $\bar{\gamma}_k$ is a simple Reeb orbit. Since u_1 is an unbranched covering, each of its puncture also converges to one of the simple Reeb orbits, $\bar{\gamma}_k$.

One also sees from (3.9) and (3.12) that C_W must have genus g and all asymptotes are simple.

Since the Reeb orbits form a two dimensional Morse-Bott family, the last statement follows from the transversality of puncture evaluation of $\bar{C}_W$ (Theorem 5.24 [15]). □

Now we look at the S-part C_{SH}.

Lemma 3.9. *Each component of C_{SH} is a trivial cylinder asymptotic to a simple Reeb orbit.*

Proof. C_{SH} has $k-2$ levels. Let $\tau_i : D_i \to SH$ be an irreducible component of first level of C_{SH}. Since C_W is connected and already has genus $\eta_\omega(A)$, D_i is of genus g and has a unique positive puncture since the domain of C_∞ is obtained by collapsing a genus g surface. Moreover, due to the asymptotic matching condition between two levels, this unique positive puncture of τ_i is asymptotic to a simple Reeb orbit since all the asymptotes of $\bar{C}_W$ are simple. Thus τ_i must be a trivial cylinder by Lemma 2.4. Similarly each component in higher level of C_{SH} must be a cylinder as well (In fact, there can only be one level of trivial cylinders in C_{SH} by the finite automorphism requirement of C_∞, but we do not need this more precise description). □

For the U-part C_U, in turn, Lemma 3.9 implies that all the positive punctures of C_U are simple due to the asymptotic matching condition between two

levels. Moreover, each component F_i is of genus 0 and has only one positive puncture, again due to the constraint $g(C_\infty) = g$. Thus each F_i is a plane with one simple positive puncture. From Lemma 2.5 and Lemma 3.8, the U-part is a union of some α- and β-planes.

Lemma 3.10. *If C_W is not a multiple cover, the U-part consists of either all α-planes or all β-planes.*

Proof. The proof is similar to [15] Lemma 7.8. As is explained in Lemma 2.5, an α-plane and a β-plane do not intersect only if they have the same asymptotic Reeb orbit. This must be the case to avoid self-intersection of the holomorphic building C_∞ which contradicts the embeddedness for C_{t_i} at some $i < \infty$.

Therefore, if the U-part has at least one α-plane and one β-plane, all planes must asymptote to the same Reeb orbit. If C_W is not a multiple cover, since the C_{SH} part consists of trivial cylinders, this is impossible by the last statement of Lemma 3.8.

□

Proof of Theorem 1.2: It is straightforward that when $n \in \mathbb{N}$ is large, under the assumption of Theorem 1.2 the multiple class nA has the following properties: $d(nA) > 0$, $GT(nA) \neq 0$, and it is represented by a connected symplectic surface with genus at least 2.

We adapt Welschinger's idea in [58] and adopt the notations in Section 3.1.2 here. Choose Darboux charts $U_i \subset W$, $i = 1, \cdots, d(nA)$, and consider $\mathcal{F}_{\{U_i\}}$ as in the paragraphs preceding Lemma 3.7. Now choose $x_i \in U_i$ and an arbitrary $J \in \mathcal{F}_{\{U_i\}}$, $A' = nA - \sum_{1 \leq i \leq d(e)} E_i$ as in Lemma 3.7. By Lemma 3.7 and the arguments in the paragraph following (3.6), there is a Baire set $\mathcal{J}_{reg}(nA) \subset \overline{\mathcal{J}} \cap \mathcal{F}_{\{U_i\}}$ such that for each $J \in \mathcal{J}_{reg}(nA)$, $(J_{\{x_i\}})_{t_j}$ is GT admissible for each j and there is a connected embedded $(J_{\{x_i\}})_{t_j}$-holomorphic curve C'_j in the class A'. Moreover, $(J_{\{x_i\}})_\infty$ is regular in the sense of SFT for the symplectic completion of $p^{-1}(W)$.

Now let us analyze the limit building C'_∞.

Notice that $-K_\omega(A') = 1 - \eta_\omega(A')$, so Lemma 3.8 could be applied. Also, from the fact that $C_{x_i} \cap U = \emptyset$ and $A' \cdot E_i = 1$, we have $C_{x_i} \cap C'_W = 1$. Therefore the W-part of C'_∞ cannot be a multiple cover. Therefore, by Lemma 3.10, C'_∞ intersects L transversally at finitely many points, where either all the local intersections are positive or all of them are negative. This implies that for some $j < \infty$, there is an embedded $(J_{\{x_i\}})_{t_j}$-holomorphic curve C'_{t_j} in the class A' with the same intersection property. Notice that C'_{t_j} intersects transversally with each C_{x_i} at one point. One then obtain the desired curve C in the class A by complex blowing down the (disjoint) exceptional curves C_{x_i}.

□

Remark 3.11. When $\kappa = -\infty$, given A in Theorem 1.2, we can actually find a symplectic surface intersecting L minimally in the class A, rather than nA for large n, if we further assume that $\eta(A) \geq 2$ and $A^2 \geq \eta(A) - 1$. Here $\eta(A)$ is the symplectic genus (see Section 4). This is because, by [33] one could achieve the non-triviality of GT invariants as long as $A^2 \geq \eta(A) - 1$. And if the class A is reduced (see Section 4 for the rational case and [33] the general case), one only needs easily verified conditions $\eta_\omega(A) \geq 2$ and $A^2 \geq \eta_\omega(A) - 1$ since $\eta_\omega(A) = \eta(A)$.

Proof of Theorem 1.1: We first deal with the case of -1 sphere C. By Proposition 3.2, there is a Baire set $\mathcal{J}_{reg}([C]) \subset \overline{\mathcal{J}}$ such that for each $J \in \mathcal{J}_{reg}([C])$, there is a unique embedded $J_{t_i}-$holomorphic sphere in the class $[C]$ for each i, and J_∞ is regular in the sense of SFT for $\overline{W}$. Notice that $d([C]) = 0$, and since C has genus 0, its W-part under neck-stretching does not admit a non-trivial unbranched cover. Therefore we can apply Lemma 3.10 as in the proof of Theorem 1.2 to produce a $J_{t_i}-$holomorphic sphere C_{t_i} intersecting L minimally. C_{t_i} is symplectic isotopic to C by the last statement of Proposition 3.2.

For a symplectic sphere C with non-negative square, we follow the strategy above by first introducing U_i and $\mathcal{F}_{U_i}$. By applying Remark 2.2 and Proposition 3.2 to M and $[C]$, there is a pair $(\tilde{J}, \tilde{\Omega} = \{x_i\})$ with $\tilde{J} \in \mathcal{F}_{\{U_i\}}$, $x_i \in U_i$, and an embedded $\tilde{J}-$holomorphic sphere $\tilde{C}$ through $\{x_i\}$ with $[\tilde{C}] = [C]$. Let $(M', J_{\{x_i\}}, \omega')$, C_{x_i}, $E_i = [C_{x_i}]$, U'_i, $\mathcal{F}'_{U_i}$, $i = 1, \cdots, d(e)$ be as in Lemma 3.7. The class $A' = [C] - \sum_{1 \leq i \leq d(e)} E_i$ is represented by an $\omega'-$symplectic -1 sphere, for instance, the proper transform of $\tilde{C}$, thus Proposition 3.2 still holds for A'.

Now apply Remark 2.2 to M' and A', then Proposition 3.2 and the arguments in the first paragraph of the present subsection imply that, there is a Baire set $\mathcal{J}_{reg}([C]) \subset \overline{\mathcal{J}} \cap \mathcal{F}_{\{U_i\}}$ with the following property: for each $J \in \mathcal{J}_{reg}([C])$, there is a unique embedded $(J_{\{x_i\}})_{t_j}-$holomorphic sphere C'_{t_j} in the class A' for each j, and $(J_{\{x_i\}})_\infty$ is regular in the sense of SFT for the symplectic completion of $p^{-1}(W)$. Moreover, C'_{t_j} intersects transversally with each C_{x_i} at one point. Now, just as in the end of the proof of Theorem 1.2, for some j, $p(C'_{t_j})$ is the desired symplectic sphere in the class A, where $p : M' \to M$ is the complex blowing down map. Moreover, $p(C'_{t_j})$ is symplectic isotopic to C by the last statement of Proposition 3.2.

□

Remark 3.12. One easily sees that the above proof works for finitely many Lagrangian spheres that are pairwisely disjoint. It is not clear to the authors whether the theorem holds when they do intersect.

On the other hand, by choosing subsequences succesively, one may push off certain symplectic configurations. In particular, the following will be used in the proof of Theorem 1.5.

Corollary 3.13. *Let L be a Lagrangian sphere in a symplectic 4-manifold (M,ω), and $D=\{A_1,\cdots,A_n\}$ a homology type of a stable spherical symplectic configuration. If each A_i pairs trivially with $[L]$. Then there is a symplectic $D-$configuration disjoint from L.*

Remark 3.14. Further, we expect to be able to deform a contractible family of symplectic spheres to be disjoint from a given Lagrangian sphere. Such a result would be useful in proving Conjecture 1.7 on the uniqueness up to symplectomorphism (see Remark 5.2 and 6.4.2). The family being contractible is necessary: as pointed out to us by R. Hind, if one takes a representative of the generator of $\pi_1(Symp(S^2,\sigma))$, the graph of this generator as a circle family of symplectic spheres in a monotone $S^2\times S^2$ cannot be isotoped away from the antidiagonal.

4 K-null spherical classes when $\kappa=-\infty$

It is in general difficult to determine whether a spherical class has a Lagrangian spherical representative. We are able to completely solve this problem for rational and ruled manifolds in Section 5.2. In this section we first derive some preliminary results.

4.1 Rational manifolds

We fix some notations: in this section M is $\mathbb{C}P^2\#n\overline{\mathbb{C}P}^2$ with $n\geq 1$. Let $\mathcal{E}$ and $\mathcal{L}$ be the sets of integral homology classes represented by smoothly embedded spheres of square -1 and -2 respectively.

An orthogonal basis $\{H,E_1,\cdots,E_n\}$ of $H_2(M;\mathbb{Z})$ is called standard if $H^2=1$ and $E_i\in\mathcal{E}$. We fix a standard basis in this section.

Let $\mathcal{K}$ be the set of symplectic canonical classes of M. For any sequence $\{\delta_i\}, i=0,...,n$ with $\delta_i=0$ or 1, let $K_{\{\delta_i\}}$ be the Poincáre dual of

$$-3H+(-1)^{\delta_1}E_1+(-1)^{\delta_2}E_2-\cdots+(-1)^{\delta_n}E_n.$$

Then $K_{\{\delta_i\}}\in\mathcal{K}$. When $\delta_i=0$ for any i, we simply denote it by K_0, i.e.

$$K_0=PD(-3H+E_1+\cdots+E_n).$$

4.1.1 $\mathcal{E},\mathcal{L}$, symplectic genus and $D(M)$

We review some facts about $\mathcal{E},\mathcal{L}$, $D(M)$ and the notion of symplectic genus.

Let $D(M)$ be the geometric automorphism group of M, i.e. the image of the diffeomorphism group of M in $Aut(H_2(M;\mathbb{Z}))$. We say two classes in $H_2(M;\mathbb{Z})$ are *equivalent* if they are related by $D(M)$.

It is shown in [33] that $D(M)$ is generated by a set of spherical reflections. For $\gamma \in H_2(M;\mathbb{Z})$ with $\gamma^2 = \gamma \cdot \gamma = \pm 1$ or ± 2, there is an automorphism $R(\gamma) \in \mathrm{Aut}(H_2(M;\mathbb{Z}))$ called the reflection along γ,

$$R(\gamma)(\beta) = \beta - \frac{2(\gamma \cdot \beta)}{\gamma \cdot \gamma}\gamma.$$

If $\gamma \in \mathcal{E}$ or $\mathcal{L}$, by Proposition 2.4 in Chapter III in [19], $R(\gamma) \in D(M)$, and we call it a spherical reflection.

Another fact is that $D(M)$ acts transitively on $\mathcal{K}$ ([36]).

To define the symplectic genus of $e \in H_2(M;\mathbb{Z})$ first introduce the subset $\mathcal{K}_e$ of $\mathcal{K}$:

$$\mathcal{K}_e = \{K \in \mathcal{K} | \text{there is a class } \tau \in \mathcal{C}_K \text{ such that } \tau \cdot e > 0\}.$$

Here $\mathcal{C}_K = \{[\omega] | \omega \text{ is a symplectic form, } K_\omega = K\}$ is the K-symplectic cone. It is shown in [36] that $\mathcal{C}_K$ is completely determined by the set of K-exceptional spherical classes

$$\mathcal{E}_K = \{E \in \mathcal{E} | K(E) = -1\}.$$

More precisely,

$$\mathcal{C}_K = \{\tau \in H^2(M;\mathbb{R}) | \tau^2 > 0, \tau(E) > 0 \text{ for any } E \in \mathcal{E}_K\}.$$

The following is a useful observation.

Lemma 4.1. *If $\xi = aH - \sum b_i E_i \in H_2(M;\mathbb{Z})$ with $a > 0$ then $K_{\{\delta_i\}} \in \mathcal{K}_\xi$.*

Proof. Notice that for any $K_{\{\delta_i\}}$, one could easily find $\tau \in \mathcal{C}_{K_{\{\delta_i\}}}$ by requiring $\tau(H) \gg 0$, but keeping the corresponding signs of E_i in τ opposite to that of $K_{\{\delta_i\}}$. Such a construction follows from the easy observation that classes in $\mathcal{E}_{K_{\{\delta_i\}}}$ are obtained by changing the corresponding signs of those in $\mathcal{E}_K$ and Theorem 4 of [36].

By possibly even enlarging $\tau(H)$ further, since $a > 0$, one could also assure that $\tau(\xi) > 0$. Therefore, $K_{\{\delta_i\}} \in \mathcal{K}_\xi$. $\square$

For $K \in \mathcal{K}_e$ define the $K-$symplectic genus $\eta_K(e)$ to be $\frac{1}{2}(K(e) + e^2) + 1$. Finally, the *symplectic genus* of class e is defined as:

$$\eta(e) = \max_{K \in \mathcal{K}_e} \eta_K(e).$$

By Lemma 3.2 in [33], $\eta(e)$ has the following basic properties:

(1) $\eta(e)$ is no bigger than the minimal genus of e, and they are both equal to $\eta_\omega(e)$ in (3.1) if e is represented by an $\omega-$symplectic surface for some symplectic form ω;

(2) Equivalent classes have the same η.

Note that in [33] these properties are stated for classes with positive square, but the proof actually covered all cases.
We have the following assertions characterizing $\mathcal{E}$ and $\mathcal{L}$ in terms of the symplectic genus, as well as the action of $D(M)$ on $\mathcal{E}$ and $\mathcal{L}$.

Proposition 4.2 ([33], Lemma 3.4, Lemma 3.6(2))**.** *For e with $e \cdot e = -1$ or -2, $\eta(e) = 0$ if and only if e is not equivalent to a reduced class.*

Moreover, for e with $e \cdot e = -1$, $\eta(e) = 0$ if and only if $e \in \mathcal{E}$, Any class in $\mathcal{E}$ is equivalent to either E_i or $H - E_i - E_j$ for some $1 \leq i, j \leq n$. If $n \neq 2$, it is equivalent to E_i.

Similarly, for e with $e \cdot e = -2$, $\eta(e) = 0$ if and only if $e \in \mathcal{L}$. Any class in $\mathcal{L}$ is equivalent to either $E_i - E_j$ or $H - E_i - E_j - E_k$ for some $1 \leq i, j, k \leq n$. If $n \neq 3$, it is equivalent to $E_i - E_j$.

Here a class $\xi = aH - \sum_{i=1}^{n} b_i E_i$ with $a \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$ is called *reduced* ([20], [28]) if

$$a \geq b_1 + b_2 + b_3.$$

4.1.2 $K-$null spherical classes and $D_K(M)$

For $K \in \mathcal{K}$ let $D_K(M)$ be the isotropy subgroup of K of the transitive action of $D(M)$ on $\mathcal{K}$. We say two classes are $K-$*equivalent* if they are related by $D_K(M)$.

By definition 1.3, $\xi \in H_2(M;\mathbb{Z})$ is a K-null spherical class if $\xi \in \mathcal{L}$ and $K(\xi) = 0$. Hence the set of $K-$null spherical classes is denoted by $\mathcal{L}_K$.

We now study the interactions of $\mathcal{L}_K$ and $D_K(M)$. Due to the transitivity of the action of $D(M)$ on $\mathcal{K}$ (c.f. [36]), we will restrict to the case $K = K_0$ without loss of generality.

First of all, if $\gamma \in \mathcal{L}_{K_0}$, then $R(\gamma) \in D_{K_0}(M)$, and we call it a K_0-*twist*.

Secondly, notice that D_{K_0} acts on $\mathcal{E}_{K_0}$ and $\mathcal{L}_{K_0}$.

To go further, we need to understand $\mathcal{E}_{K_0}$. It is clear that $E_i \in \mathcal{E}_{K_0}$. Moreover, for any symplectic form ω with $K_\omega = K_0$, the GT invariant of H or any $E \in \mathcal{E}_{K_0}$ is non-trivial. By the positivity of intersection, we have

Lemma 4.3. *Suppose $\xi = aH - \sum b_i E_i$ is in $\mathcal{E}_{K_0}$, then $a \geq 0$ and $b_i \geq 0$. If $a = 0$, then $\xi = E_i$ for some i.*

It is clear that reflections $R(E_i - E_j)$ and $R(H - E_i - E_j - E_k)$ are K_0-twists. With this understood, we see that Proposition 1.2.12 in [46] can be stated as follows,

Proposition 4.4. *Any class in $\mathcal{E}_{K_0}$ can be transformed to either E_i or $H - E_i - E_j$ for some $1 \le i, j \le n$ via K_0-twists. If $n \ne 2$, it is K_0-equivalent to E_i via K_0-twists.*

As a consequence, we have

Corollary 4.5. *Suppose $b^-(M) = n \ge 2$. If $\{E_i'\}_{i=1}^k$, $k \le n-2$, is an orthogonal subset of $\mathcal{E}_{K_0}$, then there exists $\phi \in D_{K_0}(M)$, generated by K_0-twists, such that $\phi(E_i') = E_i$, $1 \le i \le k$.*

Proof. The statement is vacuous if $n \le 2$ and easily verified for $n = 3$. We apply induction on n. From Proposition 4.4, there exists $\tilde{\phi} \in D_{K_0}(M)$ such that $\tilde{\phi}(E_1') = E_i$. One then further compose the K_0-twist $f = R(E_i - E_1)$ so that E_1' is eventually sent to E_1. Noting that

$$f(\tilde{\phi}(E_i')) \cdot E_1 = f(\tilde{\phi}(E_i')) \cdot f(\tilde{\phi}(E_1')) = E_i' \cdot E_1' = -\delta_{1i},$$

we are reduced to the case $n-1$ by restricting our attention to the last $(n-1)$ exceptional classes (and k is reduced by 1 as well). □

Remark 4.6. Note that this is not true when $k = n - 1$. Take $n = 2$. Then $H - E_1 - E_2$ is not equivalent to E_1 or E_2 since it is characteristic but E_i is not.

Proposition 4.7. *$D_{K_0}(M)$ is generated by K_0-twists.*

Proof. For $\phi \in D_{K_0}(M)$, apply Corollary 4.5 to $\phi(E_i), 1 \le i \le n-2$, there is a K_0-twist f such that $f(\phi(E_i)) = E_i$.

Consider $F_{n-1} = f(\phi(E_{n-1}))$ and $F_n = f(\phi(E_n))$. F_{n-1} and F_n are orthogonal to $E_i, 1 \le i \le n-2$, since $f(\phi(E_j)) \cdot E_i = E_j \cdot E_i = 0$ for $i \le n-2$ and $j > n-2$. It is easy to see that the only such classes in $\mathcal{E}_{K_0}$ are $H - E_{n-1} - E_n, E_{n-1}, E_n$. Since $F_{n-1} \cdot F_n = 0$, it has to be that $\{F_{n-1}, F_n\} = \{E_{n-1}, E_n\}$. By composing f with the K_0-twist $R(E_n - E_{n-1})$ if necessary, one obtains the desired inverse of ϕ generated by K_0-twists, which means ϕ is also generated by K_0-twists.

□

We now prove an analogue of Proposition 4.4 for $\mathcal{L}_{K_0}$. We start with

Lemma 4.8. *Suppose $\xi = aH - \sum b_i E_i \in H_2(M;\mathbb{Z})$ is in $\mathcal{L}_{K_0}$, If $a > 0$ then $\eta(\xi) = \eta_{K_0}(\xi)$ and $b_i \ge 0$.*

Proof. For any $\xi \in \mathcal{L}_{K_0}$, $\eta_{K_0}(\xi) = 0$ and the minimal genus is 0 as well. By Lemma 4.1, if $\xi = aH - \sum b_i E_i \in H_2(M;\mathbb{Z})$ with $a > 0$, then $\eta_{K_{\{\delta_i\}}}(\xi)$ is defined. Recall from the minimal genus assumption and the fact that the symplectic genus is no bigger than the minimal genus, $0 = \eta_{K_0}(\xi) \ge \eta_{K_{\{\delta_i\}}}(\xi)$ for any choice of $\{\delta_i\}$. But this holds only if $b_i \ge 0$ for all i, hence the conclusion follows. □

Following Evans [15], we make the following definition.

Definition 4.9. A class is called *binary* if it is of the form $E_i - E_j$, and *ternary* if it is of the form $H - E_i - E_j - E_k$, $1 \leq i, j, k \leq n$.

Clearly, binary and ternary classes are in $\mathcal{L}_{K_0}$. In the rest of our paper, we denote $R(H - E_i - E_j - E_k)$ by Γ_{ijk} for short.

Proposition 4.10. *For $\xi \in \mathcal{L}_{K_0}$, either ξ is K_0-equivalent to a binary or ternary class. Further, if either $n \neq 3$, or $n = 3$ but $\pm\xi \neq H - E_1 - E_2 - E_3$, then ξ is K_0-equivalent to the binary class $E_1 - E_2$.*

Proof. Let $\xi = aH - \sum b_i E_i$. When $a = 0$ it is easy to conclude that ξ is binary. Let r be the number of nonzero b_i. An easy calculation verifies the case when $r \leq 3$. Thus we assume $r > 3$ with $a > 0$ by possibly reversing the signs of ξ (simply do a reflection with respect to ξ). By Lemma 4.8, we may assume that $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$.

Now we write down the reflection Γ_{123} explicitly:

$$\Gamma_{123}(\xi) = (2a - b_1 - b_2 - b_3)H - \sum c_i E_i,$$

where $c_i = b_i$ for $i > 3$.

If $2a - b_1 - b_2 - b_3 < 0$, consider the class $-\Gamma_{123}(\xi) \in \mathcal{L}_{K_0}$. In this case, the leading coefficient of $-\Gamma_{123}(\xi)$ is bigger than 0. However, since $r > 3$, one must have $-c_r = -b_r < 0$, a contradiction to Lemma 4.8. Thus, $2a - b_1 - b_2 - b_3 \geq 0$

Moreover, from Lemma 4.2, ξ is not reduced hence one must have $b_1 + b_2 + b_3 > a$. Combining these facts, we have

$$0 \leq 2a - b_1 - b_2 - b_3 < a.$$

Also notice that $\Gamma_{123}(\xi)$ verifies all conditions of Lemma 4.8, thus $c_i > 0$ still holds. One could then repeat the above process and use induction on the coefficient $H \cdot \xi$ until $r \leq 3$ or $a = 0$.

□

Remark 4.11. The algorithm reducing a K-null spherical classes is also valid for exceptional classes. In this case, one gets an explicit K_0-equivalence from an exceptional class to E_i when $n \geq 3$ or possibly $H - E_1 - E_2$ when $n = 2$. This is also used in [46].

4.1.3 $(K, \alpha)-$null spherical classes and $D_{K,\alpha}(M)$

In this section we fix a class α in the $K-$symplectic cone $\mathcal{C}_K$.

Definition 4.12. A $(K, \alpha)-$*null spherical class* is a $K-$null spherical class which pairs trivially with α.

Reflections $R(\xi)$, for ξ a $(K,\alpha)-$null spherical class, are called $(K,\alpha)-$*twists*. We also define $D_{K,\alpha}(M)$ to be the subgroup of $D_K(M)$ preserving α. One has the following easy observation:

Lemma 4.13. *If $\phi \in D_K$ then*

- *ϕ induces a bijection from $\mathcal{L}_{K,\alpha}$ to $\mathcal{L}_{K,\phi^{-1}(\alpha)}$.*
- *$f \to \phi^{-1} \circ f \circ \phi$ defines an isomorphism from $D_{K,\alpha}$ to $D_{K,\phi^{-1}(\alpha)}$ taking $R(\xi)$ to $R(\phi(\xi))$.*
- *α has a positive lower bound on $\mathcal{E}_K$ which is attained by some K-exceptional class.*

The third assertion is a consequence of Gromov compactness and the well-known fact that, for any $E \in \mathcal{E}_K$, $GT(E) \neq 0$ with respect to any symplectic form ω representing α. We are now ready to prove the following:

Proposition 4.14. *$D_{(K_0,\alpha)}$ is generated by $(K_0,\alpha)-$twists.*

Proof. We will use induction on n and a trick due to Martin Pinsonnault [47]. For $n \leq 3$ this is easy to verify directly by listing all exceptional classes.

If $n \geq 3$ choose $\{E_i'\}_{i=1}^{n-2} \subset \mathcal{E}_{K_0}$ such that E_1' has minimal α-area, and E_i' has minimal α-area among exceptional classes orthogonal to E_j for all $j < i$. By Corollary 4.5, there is $\psi \in D_{K_0}(M)$ such that $\psi(E_i') = E_i$. By Lemma 4.13 we can assume that $E_i' = E_i$, so that among the basis elements $\{H, E_1, \cdots, E_n\}$, $E_1, \cdots, E_{n-2}$ enjoys the above minimality property.

Let $f \in D_{(K_0,\alpha)}$. If one could find a series of $(K_0,\alpha)-$twists such that their composition ϕ satisfies $\phi \circ f(E_1) = E_1$, one can then include ϕ^{-1} into our decomposition of f. Since E_1 is orthogonal to $\phi \circ f(E_i)$ for $i \neq 1$, one can then use induction on these classes. Therefore it suffices to look for such a ϕ in the rest of the proof.

Notice first that

$$\alpha(H - E_i - E_j - E_k) \geq 0, \ \ i > j > k. \tag{4.1}$$

This is clear from the construction: since the K_0-exceptional class $(H - E_i - E_j) \cdot E_l = 0$, for all $l < k$ and $k \leq n-2$, we have $\alpha(H - E_i - E_j) \geq \alpha(E_k)$

Assume $f(E_1) = aH - \sum b_{r_i} E_{r_i}$. Notice that $f(E_1) \in \mathcal{E}_{K_0}$ and $\alpha(f(E_1)) = \alpha(E_1)$. If $a = 0$ then $f(E_1) = E_k$ for some k and $E_1 - E_k \in \mathcal{L}_{K_0,\alpha}$. In particular, $R(E_1 - E_k) \in D_{K_0,\alpha}$ and we can choose $\phi = R(E_1 - E_k)$.

If $a \neq 0$, by Lemma 4.3, $a > 0$ and $b_i \geq 0$. Suppose $b_{r_1} \geq b_{r_2} \geq \cdots \geq b_{r_n} \geq 0$. Now apply $\Gamma_{r_1 r_2 r_3}$,

$$\Gamma_{r_1 r_2 r_3}(f(E_1)) = f(E_1) + (a - b_{r_1} - b_{r_2} - b_{r_3})(H - E_{r_1} - E_{r_2} - E_{r_3})$$

From Lemma 4.2, $a - b_{r_1} - b_{r_2} - b_{r_3} < 0$. By (4.1), $\alpha(H - E_{r_1} - E_{r_2} - E_{r_3}) \geq 0$, thus

$$\alpha(E_1) = \alpha(f(E_1)) \geq \alpha(\Gamma_{r_1 r_2 r_3}(f(E_1))).$$

By the choice of E_1, we must have $\alpha(H - E_{r_1} - E_{r_2} - E_{r_3}) = 0$. This means that $H - E_{r_1} - E_{r_2} - E_{r_3} \in \mathcal{L}_{K_0,\alpha}$ and $\Gamma_{r_1 r_2 r_3} \in D_{K_0,\alpha}(M)$.

Now from Remark 4.11, by repeating the above operations we eventually have an equivalence between E_1 and E_k for some k. Denote their composition to be $\tilde{\phi}$.

If $k = 1$ we let $\phi = \tilde{\phi}$. If $k \neq 1$, then $\alpha(E_k) = \alpha(E_1)$ and we let $\phi = R(E_1 - E_k) \circ \tilde{\phi}$.

□

4.2 Irrational ruled manifolds

It is clear that a minimal symplectic irrational ruled manifold does not admit any Lagrangian spheres. Thus, in this subsection, $M = (\Sigma_h \times S^2) \# n\overline{\mathbb{C}P}^2$. Any non-minimal genus h ruled manifold is of this form. Define $\mathcal{E}, \mathcal{L}, \mathcal{K}, D(M)$ as above. For $K \in \mathcal{K}$ also define $D_K(M), \mathcal{E}_K, \mathcal{L}_K$ and $K-$null spherical class as above.

A standard homology basis consists of $\{T, F, E_1, \cdots, E_n\}$, with the following algebraic properties:

$$T \cdot F = 1, \quad T^2 = F^2 = T \cdot E_i = F \cdot E_i = 0, \quad E_i^2 = -1, \quad 1 \leq i \leq n. \tag{4.2}$$

Geometrically, T is represented by a surface with genus h, F the class of a fiber, and $\{E_i\}$ a maximal collection of orthogonal exceptional classes in $\mathcal{E}$. The standard canonical class is then $K_0 = PD(-2T + (2h-2)F + \sum E_i)$.

$D(M)$ is characterized as the subgroup of $Aut(H_2(M; \mathbb{Z}))$ preserving F up to sign ([19]). Due to the transitive action of $D(M)$ on $\mathcal{K}$ shown in [36], we may again restrict to the case $K_\omega = K_0$.

Lemma 4.15. $\mathcal{E}_{K_0} = \{E_i, F - E_i, i = 1, ..., n\}$.

$\mathcal{L}_{K_0} = \{\pm(F - E_i - E_j), \pm(E_i - E_j), 1 \leq i < j \leq n\}$.

Proof. First of all, if $\xi = aT + bF + \sum c_i E_i$ is represented by a sphere, then $a = 0$. This follows from the fact that a sphere does not have a nonzero degree map to a positive genus curve.

With this understood, it is easy to determine $\mathcal{E}_{K_0}$ and $\mathcal{L}_{K_0}$ using (4.2).

□

For $\alpha \in \mathcal{C}_{K_0}$ we define $D_{K_0,\alpha}$, $(K_0, \alpha)-$twist as before.

Lemma 4.16. *$D_{K_0,\alpha}$ is generated by (K_0, α) twists.*

Proof. As in the rational manifold case, we do induction on $n = b^-(M) + 1$.

When $n = 1$, since $\phi(F) = \pm F$, it is easier to see that D_{K_0}, and hence $D_{K_0,\alpha}$, is trivial.

In general when $n \geq 2$, for $\phi \in D_{K_0,\alpha}$ we consider its action on $\mathcal{E}_{K_0}$. Let E be the exceptional class with minimal α area, the induction is immediate if $\phi(E) \cdot E = 0$, in which case we simply compose ϕ with the $(K_0, \alpha)-$ twist $R(E - \phi(E))$ to reduce to a lower n case.

Otherwise, $\phi(E) = F - E$ by Lemma 4.15. In this case $2\alpha(E) = \omega(F)$. Since two classes A and $F - A$ are either both in $\mathcal{E}_{K_0}$ or neither, the minimality of $\alpha(E)$ forces all other exceptional spheres to have the same area as E. Since $n \geq 2$, it is clear that one could send $F - E$ back to E via a composition of $(K_0, \alpha)-$twists, for example, the $(K_0, \alpha)-$ twist $R(E' - E)$ followed by $R(F - E' - E)$, where E' is another exceptional standard basis element orthogonal to E. Again we are able to reduce to a lower n case.

□

5 Lagrangian spherical classes when $b^+ = 1$

Theorem 1.2 allows us to effectively apply a Lagrangian-relative version of inflation procedure in this section. Together with Proposition 4.10, this enables us to classify Lagrangian spherical classes in symplectic 4-manifolds with $\kappa = -\infty$. We also give the proof of Theorem 1.8 in 5.3.

5.1 Lagrangian relative inflation

The inflation procedure was first introduced by Lalonde [29] and proved useful in many fundamental problems in symplectic geometry (see [30] for example).

The inflation construction in [29], together with Theorem 1.2, gives

Lemma 5.1 (**Inflation Lemma**). *Let L be a Lagrangian sphere in a symplectic 4-manifold with $b^+ = 1$. Let A be a class in $H_2(M; \mathbb{Z})$ satisfying the condition in Theorem 1.2. Assume also that $A \cdot [L] = 0$. Then there is a closed form ρ on M in class $PD(A)$ supported away from L so that*

$$\beta_t = \omega + t\rho, \quad t \geq 0,$$

is symplectic. In particular, L remains Lagrangian for any β_t.

The proof is straightforward: note in [29], ρ is supported near a symplectic surface in class A. Therefore, if such a symplectic surface is disjoint from the given Lagrangian sphere L, L remains Lagrangian in the course of the inflation procedure. Now Theorem 1.2 provides the desired symplectic surface.

We first apply Lemma 5.1 to study symplectic ball embeddings in the complement of a Lagrangian sphere. P. Biran and O. Cornea studied Lagrangian relative embeddings in [10] (called *mixed packing* there), where the size of maximal ball embeddings is found in some cases.

In our case of a Lagrangian sphere L in a symplectic 4-manifold with $b^+ = 1$, Lemma 5.1 enables us to show that packing problems in the complement of L can often be answered in the same way as for the ordinary packing problems. Here is one example. Biran showed in [8] that in any closed symplectic 4-manifold with an integral symplectic form, the symplectic packing problem is stable via inflation on a Donaldson hypersurface. For a symplectic 4-manifold (M, ω) with $b^+ = 1$ and ω integral, the class $n[\omega]$ for n large satisfies the conditions in Theorem 1.2 for an arbitrary given Lagrangian sphere. Thus Lemma 5.1 can be applied to such a class and hence Biran's stability result is also valid for $M \backslash L$.

Remark 5.2. It would be useful to prove the following parameterized version of Lemma 5.1, which would be the analogue of Lemma 1.1 in [42]: Given a path ω_t, $0 \leq t \leq 1$, of symplectic forms on M with $b^+ = 1$ and a sphere L Lagrangian for each ω_t. Let A be a class in $H_2(M; \mathbb{Z})$ satisfying the conditions in Theorem 1.2. Assume also that $A \cdot [L] = 0$. Then there is a path ρ_t of closed forms on M in class $PD(A)$ supported away from L so that

$$\beta_t = \omega_t + \kappa(t)\rho_t, \quad 0 \leq t \leq 1,$$

is symplectic whenever $\kappa(t) \geq 0$. In particular, L remains Lagrangian for any β_t.

Lemma 1.1 in [42] is used to show that the ball embedding space

$$E_{\bar{\lambda},k}(M, \omega) = \{\psi | \psi : \coprod_{i=1}^{k} (B_4(\lambda_i), \omega_{std}) \hookrightarrow (M, \omega)\}$$

with $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$, is connected. Substituting Lemma 1.1 in [42] by its L relative version as above in appropriate places, we would be able to obtain the connectedness of the relative ball embedding space.

5.2 Existence of Lagrangian spheres

In this subsection we present the proof of Theorem 1.4. We begin with some general discussions of Lagrangian spheres in a non-minimal symplectic 4-manifold with $b^+ = 1$.

5.2.1 Non-minimal 4-manifolds with $b^+ = 1$ and $\kappa \geq 0$

We begin with the following two persistence results.

Lemma 5.3. *Let (M,ω) be a symplectic 4-manifold with $b^+(M)=1$, $[\omega]\in H^2(M;\mathbb{Q})$. Let $(\overline{M},\overline{\omega})$ be the one point blow up of (M,ω) with size a, and $\iota: H_2(M;\mathbb{Z})\to H_2(\overline{M},\mathbb{Z})$ the canonical injection. If $L\subset(M,\omega)$ is a Lagrangian sphere, then there is a Lagrangian sphere in $(\overline{M},\overline{\omega})$ in the class $\iota([L])$.*

Proof. By the uniqueness of blow ups (Corollary 1.3 in [42]), we can place the ball of size a anywhere in (M,ω). If the ball is disjoint from L, we are done. Otherwise, first choose a ball of size $a'<a$ and disjoint from L, we obtain a blow up $(\overline{M},\overline{\omega}')$ with a Lagrangian $\bar{L}$ from L. Let $p:\overline{M}\to M$ be a topological blow down map which contracts the exceptional sphere. Consider the class $\beta_{l,\delta}=l([p^*\omega]-(a+\delta)PD(E))$ for $\delta>0$. Clearly, $\beta_{l,\delta}([\bar{L}])=0$. Since the $K_{\overline{\omega}}-$symplectic cone $\mathcal{C}_{K_{\overline{\omega}}}$ is open, we can assume that $\beta_{l,\delta}$ is in $\mathcal{C}_{K_{\overline{\omega}}}$ by choosing δ small. If $a+\delta$ is further assumed to be a rational number, then there exists $l\in\mathbb{Z}^+$ such that $\beta_{l,\delta}$ satisfies the conditions in Lemma 5.1. Applying Lemma 5.1 to such a $\beta_{l,\delta}$ and $\bar{L}$, we find that $\bar{L}$ remains Lagrangian in $(\overline{M},\overline{\omega}'')$, where $\overline{\omega}''$ is a symplectic form in the class $[p^*\omega]-aPD(E)$ up to a rescale. The proof is finished by again invoking the uniqueness of blow ups. □

If E is the class of the exceptional sphere, this lemma can be viewed as the persistence of Lagrangian spheres under a symplectic deformation on $\overline{M}$ in the E direction, which can also be proved via the inflation construction along a symplectic surface with negative self intersection as in [38].

Lemma 5.4. *Let $(\overline{M},\overline{\omega})$ be a symplectic 4-manifold with $b^+(\overline{M})=1$, $[\overline{\omega}]\in H^2(\overline{M};\mathbb{Q})$. If there are two orthogonal exceptional classes E_1, $E_2\in\mathcal{E}_\omega$ with equal symplectic area a, then there is a Lagrangian sphere in the binary class E_1-E_2.*

Proof. Let us first consider a local model: the two point blow up of a standard ball with equal size $t>0$. This can be identified with the complement of a line in $\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$ with a symplectic form τ with $[\tau]=PD(H-tE_1-tE_2)$. Notice that $(\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2,\tau)$ is symplectomorphic to a one point blow up of a monotone $S^2\times S^2$ with size $1-2t$. If we apply Lemma 5.3 to the antidiagonal L_a in this monotone $S^2\times S^2$, we find a Lagrangian sphere in $(\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2,\tau)$ in the class $E_1-E_2=\iota([L_a])$. In addition, such a Lagrangian sphere can be made disjoint from an embedded H-class sphere in $\mathbb{C}P^2\#2\overline{\mathbb{C}P}^2$ by Theorem 1.1. We therefore obtain a Lagrangian sphere in our local model.

In general, let (M,ω) be obtained by symplectically blowing down two disjoint spheres in E_1 and E_2 in $(\overline{M},\bar{\omega})$ and adopt notations in Lemma 5.3. We shrink both balls corresponding to E_1 and E_2 to size $\epsilon\ll 1$. By the uniqueness of ball-embeddings (in case of absence of a Lagrangian sphere, see Remark 5.2), we may place the two tiny balls V_1 and V_2 in a Darboux chart. Our local model analysis above ensures that there is a Lagrangian sphere L in the blow-up of the

chart around V_1 and V_2. Consider the class $B_b = PD(p^*\omega) - bE_1 - bE_2$ where b is a positive rational number slightly larger than $a = \overline{\omega}(E_i)$, $i = 1, 2$. Since the $K_{\overline{\omega}}-$symplectic cone $\mathcal{C}_{K_{\overline{\omega}}}$ is open, we can further assume that $PD(B_b)$ is in $\mathcal{C}_{K_{\overline{\omega}}}$. Clearly, $B_b \cdot (E_1 - E_2) = 0$. Thus for some large integer l_b, $l_b B_b$ satisfies the conditions in Lemma 5.1. Now the conclusion follows from inflating along a symplectic surface in class B_b as in the proof of Lemma 5.3.

□

Corollary 5.5. *Suppose (M, ω) is a minimal symplectic manifold with $b^+ = 1$, $[\omega] \in H^2(M, \mathbb{Q})$. Suppose $(\overline{M}, \bar{\omega})$ is a k point symplectic blow-up of (M, ω) with $E_i, i = 1, ..., k$, the corresponding exceptional class, and the canonical injective map is denoted as: $\iota : H_2(M; \mathbb{Z}) \to H_2(\overline{M}; \mathbb{Z})$. Then $\xi \in H_2(\overline{M}; \mathbb{Z})$ is a Lagrangian spherical class if*

(1) either $\xi \in Im(\iota)$ and $\iota^{-1}(\xi)$ is Lagrangian spherical,

(2) or $\xi = E_i - E_j$ for some i, j, i.e. ξ is binary, and $\omega(\xi) = 0$.

If $\kappa(M) \geq 0$, these are the only Lagrangian spherical classes of $(\overline{M}, \overline{\omega})$.

Proof. (1) and (2) follow directly from Lemmas 5.3 and 5.4 respectively.

To show these are the only Lagrangian spherical classes when $\kappa(M) \geq 0$, suppose $\xi = \xi' - \sum_{i=1}^{k} a_i E_i$ is represented by a Lagrangian sphere $\bar{L}$, where $\xi' \in Im(\iota)$.

If $a_i = 0$ for all i, then apply Theorem 1.1 to find disjoint exceptional spheres in the classes E_i, which are also disjoint from $\bar{L}$. This shows that ξ' is a Lagrangian spherical class of (M, ω).

Now assume some $a_i \neq 0$. The reflection $R(\xi)$ thus sends E_1 to $a\xi' - \sum_{i>1} a_i E_i - (a_1^2 - 1)E_1$. Such a class is an exceptional class of $(\overline{M}, \bar{\omega})$. However, from the uniqueness of the minimal model for symplectic manifolds with $\kappa \geq 0$ ([40]), $a\xi' - \sum_{i>1} a_i E_i - (a_1^2 - 1)E_1 = E_j$ for some j. This shows $\xi' = 0$ and ξ is indeed binary.

□

5.2.2 Rational manifolds

Proof of Theorem 1.4, rational manifold case: The case of $S^2 \times S^2$ is well-known and so we focus on blow-ups of $\mathbb{C}P^2$ below.

Due to the transitive action of $D(M)$ on $\mathcal{K}$ mentioned in Section 4, and using definition 4.12, we are reduced to prove the following Proposition.

Proposition 5.6. *Suppose $M = \mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ with $\{H, E_1, \cdots, E_n\}$ a standard basis, and ω is a symplectic form with $K_\omega = K_0 = PD(-3H + E_1 + \cdots + E_n)$. Then $\xi \in H_2(M; \mathbb{Z})$ is represented by a Lagrangian sphere if and only if ξ is $(K_0, [\omega])-$null spherical.*

Proof. The conditions are clearly necessary. In the case $n = 2$, up to sign, the only K_0−null spherical class is the binary class $\xi = E_1 - E_2$. And if ξ is $(K_0, [\omega])$−null spherical, then E_1 and E_2 must have equal symplectic area. Thus the existence of a Lagrangian sphere has been argued in the first paragraph of Lemma 5.4.

Let us then suppose that $n > 3$. One notices that in this case ξ can also be assumed to be binary. This is because, from Proposition 4.7, there is a self-diffeomorphism ϕ of M, which induces a K_0-twist on homology and sends ξ to a binary class, and we could just consider $\phi_*(\xi)$ in $(M, (\phi^{-1})^*\omega)$. Without loss of generality we could further assume $\xi = E_1 - E_2$. If $\omega(\xi) = 0$, then, up to scaling, $PD([\omega]) = 3H - \sum b_i E_i$ with $b_1 = b_2 = b > 0$.

Blowing down a collection of disjoint exceptional spheres in the classes E_i with $i \geq 3$, we obtain $M' = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ with a symplectic form ω' in the class $[\omega] = PD(3H - bE_1 - bE_2)$. As just shown, there is a Lagrangian sphere $L \subset (M', \omega')$ in the class $E_1 - E_2$. Now apply Lemma 5.3 to obtain the desired Lagrangian sphere back in (M, ω) by performing $n - 2$ blow-ups.

Finally let us suppose that $n = 3$. A K_0−null spherical class is either binary or the ternary class $\xi = H - E_1 - E_2 - E_3$. The binary case can be treated in the same way as in the case $n > 3$. So let us assume that $\xi = H - E_1 - E_2 - E_3$. Let $(\bar{M}, \bar{\omega})$ be a one point blow up of (M, ω), E_4 the new exceptional class, and ι the canonical map. Notice that $b^-(\bar{M}) = 4$ and $\iota(\xi)$ is $(K_0, [\bar{\omega}])$−null spherical, thus there is a Lagrangian $\bar{L} \subset (\bar{M}, \bar{\omega})$ in the class $\iota(\xi)$. By applying Theorem 1.2 to $\bar{L}$ and E_4, we conclude the proof by blowing down an exceptional sphere in class E_4 disjoint from $\bar{L}$.

□

Now the proof of Theorem 1.4 in the rational manifold case is complete. □

5.2.3 Irrational ruled manifolds

Proof of Theorem 1.4, irrational ruled manifold case: Similar to the rational case, it reduces to the following statement. □

Proposition 5.7. *Suppose $M = (\Sigma_h \times S^2) \# n\overline{\mathbb{C}P}^2$ with $\{T, F, E_1, \cdots, E_n\}$ a standard basis, and ω is a symplectic form with $K_\omega = K_0 = PD(-2T(2h - 2)F + E_1 + \cdots + E_n)$. Then $\xi \in H_2(M; \mathbb{Z})$ is represented by a Lagrangian sphere if and only if ξ is $(K_0, [\omega])$−null spherical.*

Proof. We use the cut and paste procedure in [40] to reduce it to the rational manifold case.

We can view (M, ω) as a symplectic genus 0 Lefschetz fibration over Σ_h with n reducible fibers, each consisting of a pair of exceptional spheres in the classes E_i and $F - E_i$. Denote the projection by π and the image of the reducible

fibers by B. View Σ_h as assembled from a $4h$-sided polygon with the vertices going to $x_0 \in \Sigma_h$, the edges going to a $2h-$wedge of loops Λ_h. Since B is a finite set, we can assume that $B \cap \Lambda_h = \emptyset$.

We cut M along $\pi^{-1}(\Lambda_h)$ to obtain a genus 0 Lefschetz fibration V over a two disk D with n reducible fibers. Recall from Lemmas 4.13 and 4.14 in [40] that with a symplectic deformation supported near an arbitrarily small neighborhood of x_0, (M, ω) can be assumed to be a symplectic product in a neighborhood of $\pi^{-1}(\Lambda_h)$. Therefore we can compactify (V, ω) into a genus 0 Lefschetz fibration $(\bar{V}, \bar{\omega})$ over S^2 with n reducible fibers by adding a fiber F_0.

Notice that V is diffeomorphic to $(S^2 \times D^2)\# n\overline{\mathbb{C}P}^2$, and $\bar{V}$ is diffeomorphic to $(S^2 \times S^2)\# n\overline{\mathbb{C}P}^2$=$(\mathbb{C}P^2\#\overline{\mathbb{C}P}^2)\# n\overline{\mathbb{C}P}^2$. Moreover, in the standard basis representation, F corresponds to $H - E_1$, and E_i corresponds to E_i. In particular, a $(K_0, [\omega])-$null spherical class corresponds to either $H - E_1 - E_i - E_j$ or $E_i - E_j$, $2 \leq i < j \leq n$.

We have shown there are Lagrangian spheres in $(\bar{V}, \bar{\omega})$ in these classes. What remains to prove is that there are Lagrangian spheres disjoint from the symplectic sphere F_0. This is true due to Theorem 1.1, since $[F_0] = H - E_1$ is a square 0 class, orthogonal to $H - E_1 - E_i - E_j$ and $E_i - E_j$ for any $2 \leq i < j \leq n$.

□

5.3 Homological action

We are now ready to prove Theorem 1.8.

Proof. Let (M, ω) be a symplectic 4-manifold with $\kappa = -\infty$. Further assume that a standard basis is chosen. As mentioned in the proof of Theorem 1.4, fixing the canonical class causes no loss of generality. Thus we assume that $K_\omega = K_0$.

On the one hand, if $f \in Symp(M, \omega)$, then $f_* \in D_{K_0,[\omega]}(M)$. On the other hand, Theorem 1.4 implies any $(K_0, [\omega])$-twist is realized by a Lagrangian Dehn twist. With this understood, Theorem 1.8 is simply a consequence of Proposition 4.14, Lemma 4.16, and Theorem 1.4.

□

Corollary 5.8. *If (M, ω) is monotone, the representation of the symplectic mapping class group on $H_2(M; \mathbb{Z})$, namely, the Torelli part, is $D_{K_\omega}(M)$.*

Remark 5.9. Corollary 5.5 also has its counterpart, which asserts that, when $b^+(M) = 1$ and $\kappa(M) \geq 0$, the homological action of $Symp(\overline{M}, \bar{\omega})$ is generated by the homological action of $Symp(M, \omega)$ and binary Lagrangian reflections.

It would be interesting to know whether for any minimal (M, ω) with $b^+ = 1$ and $\kappa(M) \geq 0$, the homological action of $Symp(M, \omega)$ is generated by Lagrangian Dehn twists.

6 Uniqueness of Lagrangian spheres in rational manifolds

The present section is devoted to the proof of Theorem 1.5. We begin by reviewing two basic uniqueness results of Hind for $S^2 \times S^2$ and T^*S^2.

6.1 Review of Hind's results

6.1.1 $S^2 \times S^2$ via symplectic cut

For $S^2 \times S^2$ we have the uniqueness up to isotopy in [23]:

Theorem 6.1 (Hind). *Lagrangian spheres in a monotone $S^2 \times S^2$ are unique up to Hamiltonian isotopy.*

From the connectedness of $Symp(S^2 \times S^2, \sigma \oplus \sigma)$ by Gromov [22], Theorem 6.1 is equivalent to

Proposition 6.2. *Lagrangian spheres in a monotone $S^2 \times S^2$ are unique up to symplectomorphisms.*

We here offer an argument for this weaker version of uniqueness using an idea from Hind [24] turning the Lagrangian uniqueness problem into a symplectic uniqueness problem via symplectic cut. Such an argument is useful for the uniqueness of Lagrangian $\mathbb{R}P^2$ in rational manifolds (see 6.4.1). Some preparations are in order.

Denote by A, $B \in H_2(S^2 \times S^2; \mathbb{Z})$ the classes of two product factors on $S^2 \times S^2$. Let Ω_λ be the product symplectic form $\pi_1^*\sigma + (1+\lambda)\pi_2^*\sigma$ with $\lambda > 0$. Let $\mathcal{J}_\lambda$ be the space of Ω_λ-tamed almost complex structures. The following is due to Abreu and McDuff:

Theorem 6.3 ([2], Proposition 2.1, Corollary 2.8). *Suppose $l - 1 < \lambda \leq l$, l an integer. Then $\mathcal{J}_\lambda$ admits a stratification $\{U_k\}_{0 \leq k \leq l}$ with the following properties:*

(1) For any $J \in U_k$, the class $A - kB$ is represented by a unique embedded J-holomorphic sphere;

(2) Each U_k is connected.

As a consequence, we have the following claim:

Proposition 6.4. *The space of symplectic spheres with self-intersection $-2k$ in $(S^2 \times S^2, \omega_\lambda)$ is non-empty and connected if $\lambda > k - 1$.*

Proof. A symplectic sphere with self-intersection $-2k$ is in the class $A-kB$, and it exists if and only if $\lambda > k-1$. For two such symplectic spheres $C_i, i = 0, 1$, there are almost complex structures $J_i \in U_k$ such that C_i is J_i-holomorphic for $i = 0, 1$. By Theorem 6.3 (2), there is a path J_t in U_k connecting J_0 and J_1. By Theorem 6.3 (1), there is a unique sphere C_t with self-intersection $-2k$ for each J_t. This path of symplectic spheres is continuous due to Gromov's compactness.

$\square$

Proof of Proposition 6.2: Given two Lagrangian spheres L_1, L_2 in $S^2 \times S^2$ with a monotone symplectic form ω. By Weinstein's neighborhood theorem one can fix two symplectic embeddings ϕ_1, ϕ_2: $T_r^*S^2 \to S^2 \times S^2$ for some small $r > 0$. For each i, consider the geodesic flow on S^2 with the standard round metric. By performing symplectic cut on $(S^2 \times S^2, \omega)$ along the boundary of the image of ϕ_i, we obtain a pair of $S^2 \times S^2$ for each i: one comes from $\phi_i(T_r^*S^2)$, equipped with the standard monotone symplectic form of size r; and the other one comes from the complement of $\phi_i(T_r^*S^2)$, equipped with symplectic form ω_i and a symplectic (-2)-sphere Σ_i. Clearly, $[\omega_0] = [\omega_1]$.

It follows from the uniqueness of homologous symplectic structures in [30] and Proposition 6.4, there is a symplectomorphism of pairs:

$$\iota : ((S^2 \times S^2, \omega_1), \Sigma_1) \to ((S^2 \times S^2, \omega_2), \Sigma_2),$$

where ι sends a neighborhood of Σ_1 symplectomorphically to one of Σ_2. Via symplectic sum ([21]), which is the exact inverse of symplectic cut (as pointed out by Gompf), ι leads to a symplectomorphism of pairs $\Psi : ((S^2 \times S^2, \omega), L_1) \to ((S^2 \times S^2, \omega), L_2)$. $\square$

6.1.2 T^*S^2 and the symplectic mapping class group

Further exploring the symplectic cut approach in 6.1.1, we obtain an alternative proof of Hind's Lagrangian sphere uniqueness in T^*S^2 below via Seidel's description of the compactly supported symplectomorphism group of T^*S^2.

Theorem 6.5 (Hind, [24])**.** *Lagrangian spheres in (T^*S^2, ω_{std}) are unique up to Hamiltonian isotopy.*

Proof: Via the negative Liouville flow and scaling we can isotope any Lagrangian in (T^*S^2, ω_{std}) into one in $(T_1^*S^2, \omega_{std})$. Further, via the identification $(T_1^*S^2, \omega_{std}) = (S^2 \times S^2, \omega_0) \backslash \Delta$, where ω_0 is a monotone form and Δ is the diagonal of $S^2 \times S^2$, it suffices to show the the uniqueness of Lagrangian spheres in $(S^2 \times S^2, \omega_0) \backslash \Delta$.

Given two Lagrangian spheres L_1, $L_2 \in (S^2 \times S^2, \omega_0)\backslash\Delta$, we first claim that there is $\phi \in Symp_c(T_1^*S^2, \omega_{std})$ such that $\phi(L_1) = L_2$, where $Symp_c$ denotes the compactly supported symplectomorphism group.

Without loss of generality we assume $L_2 = \bar{\Delta}$, which is the antidiagonal, corresponding in turn to the zero section of T^*S^2. By Proposition 6.2, there is $\Psi \in Symp(S^2 \times S^2, \omega_0)$, such that $\Psi(L_1) = L_2$. Ψ may not fix Δ, but notice that $\Psi(\Delta) \cap \bar{\Delta}(= L_2) = \emptyset$. Since the complement of $\bar{\Delta}$ is canonically identified with a symplectic disk bundle over the diagonal, by [25] there is a symplectic isotopy $\tilde{\Phi}_t : S^2 \to (S^2 \times S^2, \omega_0)$ fixing $\bar{\Delta}$ and connecting the two symplectic spheres $\Psi(\Delta)$ and Δ. In particular, $\tilde{\Phi}_t \circ \Psi(\Delta)$ is disjoint from $\bar{\Delta}$ for each t.

Now we extend $\tilde{\Phi}_t$ to a symplectic isotopy of a neighborhood U of $\Psi(\Delta)$ which we still denote as $\tilde{\Phi}_t$ (Ex. 3.40 in [44]), and require that $\tilde{\Phi}_t(U)$ be still disjoint from $\bar{\Delta}$ for all t. We then trivially extend $\tilde{\Psi}_t$ to $\tilde{\phi}_t$, a symplectic isotopy on a neighborhood U' of $\Psi(\Delta) \cup \bar{\Delta}$, which restricts to $\tilde{\Phi}_t$ on U and to the identity near $\bar{\Delta}$. Since $H^1(U'; \mathbb{R}) = 0$, $H^2(S^2 \times S^2, U'; \mathbb{R})$ injects into $H^2(S^2 \times S^2; \mathbb{R})$. By the argument proving Banyaga's isotopy extension theorem (see for example [44], Theorem 3.19), $\tilde{\phi}_t$ extends to a global symplectic isotopy ϕ_t of $(S^2 \times S^2, \omega_0)$, where $\phi_0 = id$, $\phi_1(L_1) = L_2$, and $\phi_1|_\Delta = id$.

Consider $\phi' = \phi_1 \circ \Psi \in Symp(S^2 \times S^2, \omega_0)$. Since ϕ' is the identity on Δ, it induces a compactly supported symplectomorphism ϕ of $(T_1^*S^2, \omega_{std})$ up to isotopy, mapping L_1 to the zero section L_2.

From Seidel's description of $Symp_c(T_1^*S^2, \omega_{std})$ in [50], $\phi = \tau^n \circ \eta_1$, where τ is the Lagrangian Dehn twist along the zero section L_2, and η_t, $t \in [0, 1]$ with $\eta_0 = id$ is a compactly supported symplectic isotopy. Now it is clear that $\tau^n \circ \eta_t(L_2)$ is a path connecting L_1 to the zero section since τ fixes the zero section. $\square$

6.2 Proof of Theorem 1.5

For $k \geq 0$ we will denote by V_k the manifold $(S^2 \times S^2)\#k\overline{\mathbb{C}P}^2$. When $k \geq 1$, $V_k = \mathbb{C}P^2\#(k+1)\overline{\mathbb{C}P}^2$. Due to Theorem 6.1 and the fact that $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ has no spheres with self-intersection -2, we only need to prove Theorem 1.5 for V_k with $k = 1, 3$, and $k = 2$ but $[L]$ not characteristic. By Proposition 4.10, we may further assume that $[L]$ is the binary class $E_1 - E_2$.

Throughout this subsection, J_0 denotes the complex structure obtained from a generic k-point complex blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$. Without loss of generality, we may assume ω is a Kähler form compatible with J_0. This follows from Proposition 4.8 in [35] that the symplectic cone is the same as the J_0-compatible cone in $H^2(V_k, \mathbb{R})$ when $k \leq 8$, as well as the uniqueness of homologous symplectic forms in [42].

To prove Theorem 1.5, we apply Theorem 1.1 and follow the approach in

[15] where the monotone case is settled. For some of the details one is referred to Section 9 of [15] and 4.2 of [17].

For the binary class $E_1 - E_2$, the following stable symplectic sphere configuration *type* (Definition 3.3) $D_{E_1-E_2}$ is introduced in [15]:

- $\{H - E_1 - E_2, H\}$ when $k = 1$,
- $\{H - E_1 - E_2, H - E_3, E_3\}$ when $k = 2$,
- $\{H - E_1 - E_2, H - E_3 - E_4, E_3, E_4\}$ when $k = 3$.

Since (V_k, J_0) is a generic blow up, it is clear that there is a J_0-holomorphic $D_{E_1-E_2}$ configuration C_0.

Lemma 6.6. *Suppose L is a Lagrangian sphere in (V_k, ω) with $k \leq 3$ and $[L] = E_1 - E_2$. Then L can be Hamiltonian isotoped off C_0.*

Proof. From Corollary 3.13, in the complement of the given Lagrangian sphere L, we can find a $D_{E_1-E_2}$-configuration C.

By Corollary 3.4, C_0 and C are symplectically isotopic. Following the proof of Theorem 9 in [17], with a small perturbation along the isotopy, we may assume the symplectic spheres in the configuration intersect ω-orthogonally during the isotopy. Thus, by the symplectic neighborhood theorem, we can extend this isotopy to a neighborhood of the configuration. From the fact that C and C_0 have trivial H^1, as in the proof of Theorem 6.5, we obtain an ambient Hamiltonian isotopy Ψ_t taking C to C_0. In particular, L is Hamiltonian isotopic to $\Psi_1(L)$ which is disjoint from C_0.

□

Proposition 6.7. *Suppose there is a Lagrangian sphere L in (V_k, ω) with $k \leq 3$ and $[L] = E_1 - E_2$. When $[\omega]$ is a rational, the complement of C_0 contains a unique Lagrangian sphere up to Lagrangian isotopy.*

Proof. By Lemma 6.6 we can assume that the Lagrangian sphere L is in the complement of C_0, so the complement of C_0 contains at least one Lagrangian sphere.

We will discuss the case $k = 3$. The cases $k = 1, 2$ are similar. Up to scaling, we can write $PD([\omega]) = aH - E_1 - E_2 - b_3E_3 - b_4E_4$ since $\omega([L]) = 0$. Further, $a > 1 + b_i$ since $\omega(H - E_1 - E_i) > 0$ for $i = 3, 4$. Rewrite

$$PD([\omega]) = (H - E_1 - E_2) + (a - 1)(H - E_3 - E_4) + (a - 1 - b_3)E_3 + (a - 1 - b_4)E_4.$$

Notice that $a, b_i \in \mathbb{Q}^+$ since $[\omega]$ is assumed to rational. Since all coefficients are rational and positive, there is a large integer l, such that $PD([l\omega])$ is represented as an positive integral combination of $\{H - E_1 - E_2, H - E_3 - E_4, E_3, E_4\}$, say, with coefficients $u, v, w, z \in \mathbb{Z}^+$.

If $C_0 = C_{H-E_1-E_2} \cup C_{H-E_3-E_4} \cup C_{E_3} \cup C_{E_4}$, consider the divisor $F = uC_{H-E_1-E_2} + vC_{H-E_3-E_4} + wC_{E_3} + zC_{E_4}$. There is a holomorphic line bundle $\mathcal{L}$ with a holomorphic section s whose zero divisor is exactly F. Take an hermitian metric and a compatible connection on $\mathcal{L}$ such that the curvature form is just $l\omega$. $\phi = -log|s|^2$ defines a plurisubharmonic function with $-d(d\phi \circ J_0) = l\omega$ on the complement U_0 of the C_0.

Notice that U_0 is the same as the complement U in Proposition 4.2.1 in [17], which is shown to be biholomorphic to the affine quadric there. The rest of the argument is exactly as in the proof of Proposition 4.2.1 in [17], reducing to Theorem 6.5, the uniqueness in (T^*S^2, ω_{std}).

Consider the finite type Stein structure $(J_0, \phi/l)$ on U_0. Define $h : \mathbb{R} \to \mathbb{R}$ to be the function $h(x) = e^x - 1$ and $\phi_h = h \circ \phi$. By Lemma 3.1 in [9] and Lemma 6 in [52], (U_0, J_0, ϕ_h) is a complete Stein manifold of finite type with Kähler form $\omega_h = -d(d\phi_h \circ J_0)$. Suppose a sublevel set $Y = \phi^{-1}[0, k]$ contains all the critical points of ϕ. View (Y, ω) as a Liouville domain, and let $(\hat{Y}, \hat{\omega})$ be its symplectic completion. By Lemma 2.1.5 in [17], (U_0, ω_h) is symplectomorphic to $(\hat{Y}, \hat{\omega})$.

Since the affine quadric Q has a complete finite type Stein structure inherited from $\mathbb{C}^3$, it follows from Lemma 2.1.6 in [17] that (U_0, ω_h) is symplectomorphic to (Q, ω_{can}). Combining all the symplectomorphisms, we find that the Liouville manifold $(\hat{Y}, \hat{\omega})$ is symplectomorphic to (T^*S^2, ω_{std}).

Given any two Lagrangian spheres L_0, L_1 in the complement of C_0, they lie in a sublevel set Y of ϕ containing all the critical points. We obtain an isotopy L_t in $(\hat{Y}, \hat{\omega})$ by Hind's Theorem 6.5. Contract the isotopy L_t into the sublevel set Y using the negative Liouville flow on $(\hat{Y}, \hat{\omega})$. The endpoints of the contracted isotopy are also connected in Y to L_0 and L_1 respectively by the positive Liouville flow. Therefore, one gets the desired Hamiltonian isotopy between L_0 and L_1 in $Y \subset U_0$.

□

Proof of Theorem 1.5: As mentioned in the beginning of this subsection, we could assume that $M = V_k$ with $k = 1, 2, 3$, ω is a Kähler form compatible with J_0, and $\xi = E_1 - E_2$.

Suppose L_0 and L_1 are two Lagrangian spheres in the class ξ. By Lemma 6.6 they are Hamiltonian isotopic respectively to two Lagrangian spheres, still denoted by L_0 and L_1, in the complement U_0 of C_0. We will show that L_0 and L_1 are Lagrangian isotopic in U_0, and hence in (V_k, ω). As argued in Theorem 6.5, this implies that L_0 and L_1 are Hamiltonian isotopic.

Again we will discuss the case $k = 3$. By rescaling the symplectic form, we could still assume the ω-area of E_1 and E_2 is rational. View (V_3, ω) as a three point blow-up of a monotone $(S^2 \times S^2, \tau)$, then as the three disjoint com-

ponents of C_0, $C_{H-E_1-E_2}, C_{E_3}, C_{E_4}$ are all exceptional, corresponding to three ball embeddings h_{12}, e_3, e_4 in $(S^2 \times S^2, \tau)$. Let $\tilde{L}_0$ and $\tilde{L}_1$ be the corresponding Lagrangians in $(S^2 \times S^2, \tau)$.

Via the correspondence of ball-embeddings and symplectic forms in the blown-up manifolds, one may deform ω to ω' near $C_{H-E_1-E_2}, C_{E_3}, C_{E_4}$ such that their ω'-areas become rational. In fact, from the continuity of ball embeddings, such a deformation can be chosen to correspond to a slightly larger ball-embeddings h'_{12}, e'_3 and e'_4 in $(S^2 \times S^2, \tau)$. Further, we may assume that the larger embedded balls are still disjoint from $\tilde{L}_0$ and $\tilde{L}_1$. And when such a perturbation is chosen small enough, J_0 is still tamed by ω' so that the configuration C_0 is still symplectic with respect to ω'.

Notice that L_0 and L_1 remain Lagrangian in (V_k, ω'). Notice also that $[\omega']$ is rational, so we have a Lagrangian isotopy between L_0 and L_1 in (V_3, ω') by Proposition 6.7. It is important to observe that such an isotopy can be chosen to lie inside the complement of the $\omega'-$symplectic configuration C_0.

In particular, the isotopy does not intersect the spheres $C_{H-E_1-E_2}, C_{E_3}, C_{E_4}$. In turn it gives rise to an isotopy between $\tilde{L}_0$ and $\tilde{L}_1$ in the complement of the images of h'_{12}, e'_3 and e'_4. Since h'_{12}, e'_3 and e'_4 are extensions of h_{12}, e_3 and e_4, the isotopy between $\tilde{L}_0$ and $\tilde{L}_1$ lie in the complement of the images of h_{12}, e_3 and e_4. Therefore it gives rise to an isotopy between L_0 and L_1 in the complement of the spheres $C_{H-E_1-E_2}, C_{E_3}, C_{E_4}$ in (V_k, ω).

□

6.3 Smooth isotopy

Proof of Theorem 1.6: By Proposition 4.10, we again assume that we are in the binary case $E_1 - E_2$. Given two Lagrangian spheres L_i, following [17], consider the classes E_j, $j \geq 3$. From Theorem 1.1, for each i, we can find a set of disjoint symplectic spheres in E_j, which are also disjoint from L_i. Applying Proposition 3.4 to these two stable spherical symplectic configurations as above, we can assume that L_i are both disjoint from a set of disjoint symplectic spheres S_i in $E_j, j \geq 3$.

Blow down S_i we obtain $(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}, \omega')$ with balls B_j disjoint from L_i. Let L_t be a Lagrangian isotopy between L_i in $(\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}, \tilde{\omega})$ from Theorem 1.5. Viewed as a smooth isotopy, we can assume that L_t is transversal to the centers x_j of B_j, thus avoiding x_j. Let $B'_j \subset B_j$ be a smaller ball not intersecting L_t. Let ϕ be a diffeomorphism from U', the complement of $\cup B'_j$ to U, the complement of $\cup B_j$, which is identity near L_i. Then $\phi(L_t)$ is a smooth isotopy between L_i in U. Blowing up at x_j by cutting B_j, we get back to (M, ω) and a smooth isotopy between L_i therein.

□

6.4 Some remarks on uniqueness

We end the paper with some discussions about uniqueness.

6.4.1 Lagrangian $\mathbb{R}P^2$

The argument in 6.1.1, with (-2)-spheres replaced by (-4)-spheres, can be used to prove that any two Lagrangian $\mathbb{R}P^2$ in $(\mathbb{C}P^2, \omega_{std})$ are symplectomorphic. From Gromov's connectedness of $Symp(\mathbb{C}P^2, \omega_{std})$ in [22], we then obtain a new proof of the following result of Hind ([24]).

Theorem 6.8 (Hind). *Any two Lagrangian $\mathbb{R}P^2$ in $\mathbb{C}P^2$ are Hamiltonian isotopic to each other.*

6.4.2 Uniqueness up to symplectomorphisms

Conjecture 1.7 states that, for any two homologous Lagrangian spheres L_1 and L_2 in a symplectic rational manifold (M, ω), there exists $\phi \in Symp_h(M, \omega)$ such that $\phi(L_1) = L_2$. It implies the disconnectedness of homologically trivial symplectormophism groups in the cases when there are non-isotopic Lagrangian spheres.

We outline a possible approach to Conjecture 1.7. One easily reduces the problem to the binary case as in the proof of Theorem 1.4. Without loss of generality, let $[L_i] = E_1 - E_2$.

For each pair (M, L_i), by Theorem 1.1, away from L_i, there is a set of disjoint (-1) symplectic spheres $C_i^l, l = 3, ..., k+1$, with $[C_i^l] = E_l$ for $l = 3, ..., k$, and $[C_i^{k+1}] = H - E_1 - E_2$. Blowing down the C_l yields two $(k+1)$-tuples of $(\tilde{M}_i, \tilde{L}_i, B_i^l)$, $i = 1, 2$, $3 \leq l \leq k+1$. Here $\tilde{M}_i$ is a symplectic $S^2 \times S^2$, $\tilde{L}_i$ a Lagrangian sphere, and B_i^l a symplectic ball corresponding to C_i^l.

By [30] there is a symplectomorphism $\Psi : \tilde{M}_1 \to \tilde{M}_2$. From Theorem 6.2, there is a symplectomorphism sending $\Psi(\tilde{L}_1)$ to $\tilde{L}_2$. Composing these two symplectomorphisms one obtains a symplectomorphism between the pairs $(\tilde{M}_i, \tilde{L}_i)$, which we still denote as Ψ. The conjectured connectedness of relative symplectic ball embedding in Remark 5.2 implies that the $k-2$ balls $\Psi(B_1^l)$ can be further displaced by an $\tilde{L}_2$-preserving Hamiltonian isotopy to the balls B_2^l. This gives a symplectomorphism between the $(k+1)$-tuples $(\tilde{M}_i, \tilde{L}_i, B_i^l)$, which in turn descends to a symplectomorphism between the pairs (M, L_i).

6.4.3 Lagrangian T^2

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